CHAPTER 5

Matrices & The Binomial Theorem
Matrices are important mathematical tools in the study of mathematics itself as well as other areas of science such as physics, chemistry, computer graphic design, analyzing relationships, and even plotting complicated dance steps and many other areas. The origins of mathematical matrices lie with the study of systems of simultaneous linear equations as we studied in the previous chapter.

**Definition 1**: A matrix is a rectangular array of real numbers, called entries, written in $m$ rows and $n$ columns.

$$
\mathbf{A} = \begin{bmatrix}
  a_{11} & a_{12} & a_{13} & \cdots & a_{1n} \\
  a_{21} & a_{22} & a_{23} & \cdots & a_{2n} \\
  \vdots & \vdots & \vdots & \ddots & \vdots \\
  a_{m1} & a_{m2} & a_{m3} & \cdots & a_{mn}
\end{bmatrix}
$$

Size (order) of the matrix $= m \times n$

The **principal diagonal** of this matrix consists of entries $a_{11}, a_{22}, a_{33}, \ldots, a_{mm}$. If the case of $m = n$ the matrix is called a **square matrix**. A matrix with one column is called a **column matrix** and a matrix with on row is called a **row matrix**.

**Example 1**: For each matrix determine the size and use a dotted line to identify the principal diagonal.

$$
\mathbf{A} = \begin{bmatrix}
  2 & -1 & 5 & 3 \\
  -2 & 3 & 4 & 0 \\
  1 & -7 & 8 & 2
\end{bmatrix}
\quad
\mathbf{B} = \begin{bmatrix}
  1 & -3 & -6 \\
  4 & -3 & 2 \\
  5 & 8 & 1
\end{bmatrix}
\quad
\mathbf{C} = \begin{bmatrix}
  2 & -1 & 5 \\
  -2 & 3 & 0
\end{bmatrix}
$$

$$
\mathbf{D} = \begin{bmatrix}
  1 & -3 \\
  4 & -3 \\
  5 & 8
\end{bmatrix}
\quad
\mathbf{E} = \begin{bmatrix}
  -2 & -1 \\
  3 & 0
\end{bmatrix}
\quad
\mathbf{F} = \begin{bmatrix}
  2 \\
  -3 \\
  1 \\
  6
\end{bmatrix}
\quad
\mathbf{G} = \begin{bmatrix}
  4 & -1 & 5 & 7
\end{bmatrix}
$$

Note that a $1 \times 1$ matrix is just a single number in brackets $\mathbf{N} = \begin{bmatrix} c \end{bmatrix} = c$
Operations with Matrices:

1. **Transpose**: The transpose of a matrix \( A \) is the matrix obtained from \( A \) by writing its rows as columns. If \( A \) is an \( m \times n \) matrix, then \( A^T \) is an \( n \times m \) matrix.

2. **Sum & Difference**: If two matrices \( A \) and \( B \) have the same size \( n \times m \), then their sum \( A + B \) or their difference \( A - B \) is obtained by adding or subtracting the corresponding entries.

3. **Scalar Multiple**: If \( A \) is a matrix and \( c \) is a number (sometimes called a scalar in this context), then the scalar multiple \( cA \) is obtained by multiplying every entry in \( A \) by \( c \).

**Example 2**: Consider the following matrices:

\[
A = \begin{bmatrix} 2 & -1 & 5 \\ -2 & 3 & 4 \\ 1 & -7 & 8 \end{bmatrix} \quad B = \begin{bmatrix} 1 & -3 & -6 \\ 4 & -3 & 2 \\ 5 & 8 & 1 \end{bmatrix} \quad C = \begin{bmatrix} 2 & -1 & 5 \\ -2 & 3 & 0 \end{bmatrix}
\]

\[
D = \begin{bmatrix} 1 & -3 & 5 \\ 4 & -3 & 8 \end{bmatrix} \quad E = \begin{bmatrix} 1 & -5 \\ 4 & -3 \\ 2 & -1 \end{bmatrix} \quad F = \begin{bmatrix} -2 & -1 \\ 3 & 0 \end{bmatrix}
\]

a) Find \( A + B \) and \( C - D \), \( E + D^T \)
b) Find $A + D$ and $E - C$.

c) Find $2A - 3B$ and $C + 2E^T$

**Product of Two Matrices:**

If $A$ has dimensions $m \times p$ and $B$ has dimensions $p \times n$, then the product $AB$ is defined, and has dimensions $m \times n$. This means that for the product $c = AB$ to be defined, the number of columns of matrix $A$ must be equal to the number of rows of matrix $B$.

Special Case: Let $A = \begin{bmatrix} a_{11} & a_{12} & a_{13} & \ldots & a_{1p} \end{bmatrix}$ be a $1 \times p$ row matrix, and $B = \begin{bmatrix} b_{11} \\ b_{21} \\ \vdots \\ b_{p1} \end{bmatrix}$ be a $p \times 1$ column matrix. Then the product $AB$ is possible and it is a $1 \times 1$ matrix (a single number) and is defined as following:

$$C = AB = \begin{bmatrix} a_{11} & a_{12} & a_{13} & \ldots & a_{1p} \end{bmatrix} \begin{bmatrix} b_{11} \\ b_{21} \\ \vdots \\ b_{p1} \end{bmatrix} = \begin{bmatrix} a_{11}b_{11} + a_{12}b_{21} + a_{13}b_{31} + \ldots + a_{1p}b_{p1} \end{bmatrix} = c_{11}$$


Example 3: Multiply the row matrix $G$ and the column matrix $F$ in example 1 above.

$$GF = \begin{bmatrix} 4 & -1 & 5 & 7 \end{bmatrix} \begin{bmatrix} 2 \\ -3 \\ 1 \\ 6 \end{bmatrix} =$$

The General Case: If $A$ is an $m \times p$ matrix and $B$ is a $p \times n$ matrix, then the product $C = AB$ is possible and has dimensions $m \times n$ and the entry $c_{ij}$ of the product is obtained by multiplying the $i$th row of $A$ by the $j$th column of $B$ as described in the special case.

Example 4: Consider the following matrices

$$A = \begin{bmatrix} 2 & -1 & 5 \\ -2 & 3 & 4 \\ 1 & -7 & 8 \end{bmatrix} \quad B = \begin{bmatrix} 1 & -3 & -6 \\ 4 & -3 & 2 \\ 5 & 8 & 1 \end{bmatrix} \quad C = \begin{bmatrix} 2 & -1 & 5 \\ -2 & 3 & 0 \end{bmatrix}$$

$$D = \begin{bmatrix} 1 & -3 & 5 \\ 4 & -3 & 8 \end{bmatrix} \quad E = \begin{bmatrix} 1 & -5 \\ 4 & -3 \\ 2 & -1 \end{bmatrix} \quad F = \begin{bmatrix} -2 & -1 \\ 3 & 0 \end{bmatrix}$$

Perform the products $BA, AB, BD, CD, DE, DB, EF, FE$
Example 5: A factory has two divisions, each of which produces both tennis and football equipments. The production levels are represented by matrix $A$.

$$
A = \begin{bmatrix}
100 & 90 & 70 & 30 \\
40 & 20 & 60 & 60
\end{bmatrix}
$$

Find the production levels when the production is increased by 17%.

Example 6: A coffee dealer buys coffee from three different countries, C1, C2, C3, each of which produces the two types of coffee, regular and premium. The amount of each kind shipped are given by the matrix

$$
A = \begin{bmatrix}
120 & 145 & 95 \\
100 & 165 & 110
\end{bmatrix}
$$

The coffee is sold for the profit per unit for each type given by the matrix

$$
B = [ 2.50, 4.75 ].
$$

Compute the product $BA$ and explain what the result means. What can you say about $AB$?
Exercise

1. For each matrix determine the size and use a dotted line to identify the principal diagonal.

\[ A = \begin{bmatrix} -3 & -2 & 4 & 1 \\ -3 & -5 & 3 & 1 \\ -4 & -7 & 6 & 2 \end{bmatrix}, \quad B = \begin{bmatrix} 3 & -1 & -5 \\ 7 & -1 & 9 \\ 8 & -6 & 3 \end{bmatrix}, \quad C = \begin{bmatrix} 3 & -2 & 4 \\ -1 & 5 & 3 \end{bmatrix} \]

Size: \ldots \ldots \ldots \quad Size: \ldots \ldots \ldots \quad Size: \ldots \ldots \ldots

2. Let \( A = \begin{bmatrix} 3 & -2 & 4 \\ -5 & 2 & -1 \\ 7 & -1 & -3 \end{bmatrix}, \quad B = \begin{bmatrix} 2 & -1 & -5 \\ 8 & -4 & 2 \\ -5 & 8 & 1 \end{bmatrix}, \quad C = \begin{bmatrix} 3 & -4 & 5 \\ -1 & 2 & 1 \end{bmatrix}, \quad D = \begin{bmatrix} 1 & -3 & 5 \\ 4 & -3 & 8 \end{bmatrix}, \quad E = \begin{bmatrix} 4 & -3 \\ 5 & 0 \end{bmatrix}. \) Find

a) \( B - A = \)

b) \( C + 2D^T = \)

c) Explain why operations \( A + D, E - C, \) and \( B - D \) cannot be performed.

3. Consider the matrices

\[ A = \begin{bmatrix} 3 & -2 & 5 \\ -1 & 5 & 4 \\ 2 & -8 & 7 \end{bmatrix}, \quad B = \begin{bmatrix} 7 & -1 & 4 \\ -2 & 3 & 2 \end{bmatrix}, \quad C = \begin{bmatrix} -2 & -5 \\ 4 & -3 \\ 5 & -1 \end{bmatrix}. \)

Find the products

a) \( BA = \)

b) \( AB = \) Not possible. Explain why?

c) \( BC = \)

5. Let \( M = \begin{bmatrix} 3 & x & -3 \\ 4 & 1 & 2 \end{bmatrix} \) and \( N = \begin{bmatrix} 7 & -6 \\ y & -1 \end{bmatrix}. \) Find \( MN = \)