Differential equations are equations containing derivatives. The following are examples of physical phenomena involving rates of change:

- Motion of fluids
- Motion of mechanical systems
- Flow of current in electrical circuits
- Dissipation of heat in solid objects
- Seismic waves
- Population dynamics

A differential equation that describes a physical process is often called a \textbf{mathematical model}.

**Example 1: Free Fall**

Formulate a differential equation describing motion of an object falling in the atmosphere near sea level.

Variables: time $t$, velocity $v$

Newton’s 2\textsuperscript{nd} Law: $F = ma = m(dv/dt)$ ← net force

Force of gravity: $F = mg$ ← downward force

Force of air resistance: $F = \gamma v$ ← upward force

Then

$$m \frac{dv}{dt} = mg - \gamma v$$

Taking $g = 9.8 \text{ m/sec}^2$, $m = 10 \text{ kg}$, $\gamma = 2 \text{ kg/sec}$, we obtain

$$\frac{dv}{dt} = 9.8 - 0.2v$$

**Sketching Direction Field**

Using differential equation and table, plot slopes (estimates) on axes below. The resulting graph is called a \textbf{direction field}.

<table>
<thead>
<tr>
<th>$v$</th>
<th>$v'$</th>
</tr>
</thead>
<tbody>
<tr>
<td>0</td>
<td>9.8</td>
</tr>
<tr>
<td>5</td>
<td>8.8</td>
</tr>
<tr>
<td>10</td>
<td>7.8</td>
</tr>
<tr>
<td>15</td>
<td>6.8</td>
</tr>
<tr>
<td>20</td>
<td>5.8</td>
</tr>
<tr>
<td>25</td>
<td>4.8</td>
</tr>
<tr>
<td>30</td>
<td>3.8</td>
</tr>
<tr>
<td>35</td>
<td>2.8</td>
</tr>
<tr>
<td>40</td>
<td>1.8</td>
</tr>
<tr>
<td>45</td>
<td>0.8</td>
</tr>
<tr>
<td>50</td>
<td>-0.2</td>
</tr>
<tr>
<td>55</td>
<td>-1.2</td>
</tr>
<tr>
<td>60</td>
<td>-2.2</td>
</tr>
</tbody>
</table>
**Direction Field Using Maple**

Sample Maple commands for graphing a direction field:

```maple
with(DEtools):
DEplot(diff(v(t),t)=9.8-v(t)/5,v(t),
t=0..10,v=0..80,stepsize=.1,color=blue);
```

When graphing direction fields, be sure to use an appropriate window, in order to display all equilibrium solutions and relevant solution behavior.

Arrows give tangent lines to solution curves, and indicate where solution is increasing & decreasing (and by how much). Horizontal solution curves are called **equilibrium solutions**. Use the graph below to solve for equilibrium solution, and then determine analytically by setting \( v' = 0 \).

Set \( v' = 0 \):
\[
\iff 9.8 - 0.2v = 0 \iff v = \frac{9.8}{0.2} \iff v = 49
\]

In general, for a differential equation of the form \( y' = ay - b \), find equilibrium solutions by setting \( y' = 0 \) and solving for \( y \):
\[
y(t) = \frac{b}{a}
\]

**Example 2: Graphical Analysis**

Discuss solution behavior and dependence on the initial value \( y(0) \) for the differential equation below, using the corresponding direction field.

\[ y' = 2 - y \]

\[ y' = 5y + 3 \]
Example 5: Mice and Owls

Consider a mouse population that reproduces at a rate proportional to the current population, with a rate constant equal to 0.5 mice/month (assuming no owls present). When owls are present, they eat the mice. Suppose that the owls eat 15 per day (average). Write a differential equation describing mouse population in the presence of owls. (Assume that there are 30 days in a month.)

Solution:

\[ \frac{dp}{dt} = 0.5p - 450 \]

Graphical Analysis for a Nonlinear Equations

Example 4:

\[ y' = y(y + 2) \]

All above examples are from the family of differential equations of the form (the First Order Differential Equations)

\[ \frac{dy}{dt} = f(t, y) \]

where \( f \) is a function of \( t \) and \( y \), and it referred to as the rate function.