Ch 2.3:  
Modeling with First Order Equations

Mathematical models characterize physical systems, often using differential equations.

**Model Construction:** Translating physical situation into mathematical terms. Clearly state physical principles believed to govern process. Differential equation is a mathematical model of process, typically an approximation.

**Analysis of Model:** Solving equations or obtaining qualitative understanding of solution. May simplify model, as long as physical essentials are preserved.

**Comparison with Experiment or Observation:** Verifies solution or suggests refinement of model.

**Example 1:**

Suppose a mouse population reproduces at a rate proportional to current population, with a rate constant of 0.5 mice/month (assuming no owls present).

Further, assume that when an owl population is present, they eat 15 mice per day on average.

The differential equation describing mouse population in the presence of owls, assuming 30 days in a month, is

\[ p' = 0.5p - 450 \]

Using methods of calculus, we solved this equation in Chapter 1.2, obtaining

\[ p = 900 + ke^{0.5t} \]

**Example 2:**

At time \( t = 0 \), a tank contains \( Q_0 \) lb of salt dissolved in 100 gal of water. Assume that water containing \( \frac{1}{4} \) lb of salt/gal is entering tank at rate of \( r \) gal/min, and leaves at same rate.

(a) Set up IVP that describes this salt solution flow process.
(b) Find amount of salt \( Q(t) \) in tank at any given time \( t \).
(c) Find limiting amount \( Q_L \) of salt \( Q(t) \) in tank after a very long time.
(d) If \( r = 3 \) & \( Q_0 = 2Q_L \), find time \( T \) after which salt is within 2% of \( Q_L \).
(e) Find flow rate \( r \) required if \( T \) is not to exceed 45 min.
At time $t = 0$, a tank contains $Q_0$ lb of salt dissolved in 100 gal of water. Assume water containing $\frac{1}{4}$ lb of salt/gal enters tank at rate of $r$ gal/min, and leaves at same rate. Assume salt is neither created or destroyed in tank, and distribution of salt in tank is uniform (stirred). Then

$$\frac{dQ}{dt} = \text{rate in} - \text{rate out}$$

Rate in: $(1/4 \text{ lb salt/gal})(r \text{ gal/min}) = (r/4) \text{ lb/min}$

Rate out: If there is $Q(t)$ lbs salt in tank at time $t$, then concentration of salt is $Q(t)/100$ gal, and it flows out at rate of $[Q(t)/100] \text{ lb/min}$. Thus our IVP is

$$\frac{dQ}{dt} = \frac{r}{4} - \frac{rQ}{100}, \quad Q(0) = Q_0$$

To find amount of salt $Q(t)$ in tank at any given time $t$, we need to solve the initial value problem

$$\frac{dQ}{dt} + \frac{rQ}{100} = \frac{r}{4}, \quad Q(0) = Q_0$$

To solve, we use the method of integrating factors:

$$\mu(t) = e^{\int \frac{r}{4}} = e^{\frac{rt}{100}}$$

$$Q(t) = e^{-\frac{r}{100}} \left[ \int \frac{r}{4} Q(t) e^{\frac{rt}{100}} dt \right] = e^{-\frac{r}{100}} \left[ 25 e^{\frac{rt}{100}} + C \right] = 25 + Ce^{-\frac{r}{100}}$$

or

$$Q(t) = 25 \left( 1 - e^{-\frac{r}{100}} \right) + Q_0 e^{-\frac{r}{100}}$$

Next, we find the limiting amount $Q_L$ of salt $Q(t)$ in tank after a very long time:

$$Q_L = \lim_{t \to \infty} Q(t) = \lim_{t \to \infty} \left( 25 + [Q_0 - 25] e^{-\frac{r}{100}} \right) = 25 \text{ lb}$$

This result makes sense, since over time the incoming salt solution will replace original salt solution in tank. Since incoming solution contains 0.25 lb salt / gal, and tank is 100 gal, eventually tank will contain 25 lb salt. The graph shows integral curves for $r = 3$ and different values of $Q_0$. 

$$Q(t) = 25 \left( 1 - e^{-\frac{r}{100}} \right) + Q_0 e^{-\frac{r}{100}}$$
Suppose \( r = 3 \) and \( Q_0 = 2Q_L \). To find time \( T \) after which \( Q(t) \) is within 2\% of \( Q_L \), first note \( Q_0 = 2Q_L = 50 \) lb, hence

\[
Q(t) = 25 + [Q_0 - 25]e^{-\eta/100} = 25 + 25e^{-0.03t}
\]

\[
255 = 25 + 25e^{-0.03T} \Rightarrow 0.02 = e^{-0.03T} \Rightarrow \ln(0.02) = -0.03T \Rightarrow T = \frac{\ln(0.02)}{-0.03} \approx 1304 \text{ min}
\]

Next, 2\% of 25 lb is 0.5 lb, and thus we solve

To find flow rate \( r \) required if \( T \) is not to exceed 45 minutes, recall from part (d) that \( Q_0 = 2Q_L = 50 \) lb, with

\[
Q(t) = 25 + 25e^{-\eta/100}
\]

and solution curves decrease from 50 to 25.5. Thus we

\[
25.5 = 25 + 25e^{-\frac{45}{100}} \Rightarrow 0.02 = e^{-0.45r} \Rightarrow \ln(0.02) = -0.45r \Rightarrow r = \frac{\ln(0.02)}{-0.45} \approx 8.69 \text{ gal/min}
\]

Since situation is hypothetical, the model is valid. As long as flow rates are accurate, and concentration of salt in tank is uniform, then differential equation is accurate description of flow process. Models of this kind are often used for pollution in lake, drug concentration in organ, etc. Flow rates may be harder to determine, or may be variable, and concentration may not be uniform. Also, rates of inflow and outflow may not be same, so variation in amount of liquid must be taken into account.

**Example 3:**

Consider a pond that initially contains 10 million gallons of fresh water. Water containing toxic waste flows into the pond at the rate of 5 million gal/year, and exits at same rate. The concentration \( c(t) \) of toxic waste in the incoming water varies periodically with time:

\[
c(t) = 2 + \sin 2t \text{ g/gal}
\]

(a) Construct a mathematical model of this flow process and determine amount \( Q(t) \) of toxic waste in pond at time \( t \).

(b) Plot solution and describe in words the effect of the variation in the incoming concentration.

Pond initially contains 10 million gallons of fresh water. Water containing toxic waste flows into pond at rate of 5 million gal/year, and exits pond at same rate. Concentration is \( c(t) = 2 + \sin 2t \text{ g/gal} \) of toxic waste in incoming water.
Assume toxic waste is neither created or destroyed in pond, and distribution of toxic waste in pond is uniform (stirred). Then

\[ \frac{dQ}{dt} = \text{rate in} - \text{rate out} \]

Rate in: \((2 + \sin 2t \text{ g/gal})(5 \times 10^6 \text{ gal/year})\). Rate out: If there is \(Q(t)\) g of toxic waste in pond at time \(t\), then concentration of salt is \(Q(t)/10^7\) gal, and it flows out at rate of \([Q(t) \text{ g/10}^7 \text{ gal}][5 \times 10^6 \text{ gal/year}]\)

Rate in: \((2 + \sin 2t \text{ g/gal})(5 \times 10^6 \text{ gal/year})\)
Rate out: \([Q(t) \text{ g/10}^7 \text{ gal}][5 \times 10^6 \text{ gal/year}] = Q(t)/2 \text{ g/yr.}\)

Then initial value problem is

\[ \frac{dQ}{dt} = (2 + \sin 2t)(5 \times 10^6) - \frac{Q(t)}{2}, \quad Q(0) = 0 \]

Change of variable (scaling): Let \(q(t) = Q(t)/10^6\). Then

\[ \frac{dq}{dt} + \frac{q(t)}{2} = 10 + 5 \sin 2t, \quad q(0) = 0 \]

To solve this initial value problem we use the method of integrating factors:

\[ \mu(t) = e^{\int 1/2 \, dt} = e^{t/2} \quad \Rightarrow \quad q(t) = e^{-t/2} \int e^{t/2} (10 + 5 \sin 2t) \, dt \]

Using integration by parts (see below) and the initial condition, we obtain after simplifying,

\[ q(t) = e^{-t/2} \left[ 20e^{t/2} - \frac{40}{17}e^{t/2} \cos 2t + \frac{10}{17}e^{t/2} \sin 2t + C \right] \quad \Rightarrow \quad q(t) = 20 - \frac{40}{17} \cos 2t + \frac{10}{17} \sin 2t - \frac{300}{17} e^{-t/2} \]

Here is the detail of the integration by parts:

\[ \int e^{t/2} \sin 2t \, dt = \left[ -\frac{1}{2} e^{t/2} \cos 2t + \frac{1}{4} \left( \int e^{t/2} \cos 2t \, dt \right) \right] \]

\[ = \left[ -\frac{1}{2} e^{t/2} \cos 2t + \frac{1}{4} \left( \frac{1}{2} e^{t/2} \sin 2t - \frac{1}{4} \int e^{t/2} \sin 2t \, dt \right) \right] \]

\[ = \left[ -\frac{1}{2} e^{t/2} \cos 2t + \frac{1}{8} e^{t/2} \sin 2t - \frac{1}{16} e^{t/2} \sin 2t \right] \]

\[ \frac{17}{16} \int e^{t/2} \sin 2t \, dt = -\frac{1}{2} e^{t/2} \cos 2t + \frac{1}{8} e^{t/2} \sin 2t + C \]

\[ \int e^{t/2} \sin 2t \, dt = -\frac{8}{17} e^{t/2} \cos 2t + \frac{2}{17} e^{t/2} \sin 2t + C \]

\[ 5 \int e^{t/2} \sin 2t \, dt = -\frac{40}{17} e^{t/2} \cos 2t + \frac{10}{17} e^{t/2} \sin 2t + C \]
A graph of solution along with direction field for differential equation is given below. Note that exponential term is important for small $t$, but decays away for large $t$. Also, $y = 20$ would be equilibrium solution if not for $\sin(2t)$ term.

Amount of water in pond controlled entirely by rates of flow, and none is lost by evaporation or seepage into ground, or gained by rainfall, etc. Amount of pollution in pond controlled entirely by rates of flow, and none is lost by evaporation, seepage into ground, diluted by rainfall, absorbed by fish, plants or other organisms, etc. Distribution of pollution throughout pond is uniform.