# Table of Contents

Table of Contents .......................................................... 2

Unit 1 – Consumer Mathematics ................................................. 4
  Section 1 – Percent Change and Taxes .................................. 4
  Section 2 – Interest ............................................................ 10
  Section 3 – Consumer Loans .............................................. 16
  Section 4 - Annuities .......................................................... 21
  Section 5 – Amortization (Optional) .................................. 24
  Consumer Mathematics Review ........................................... 28

Unit 2 – Sets ........................................................................ 33
  Section 1 – The Language of Sets ...................................... 33
  Section 2 – Comparing Sets .............................................. 36
  Section 3 – Set Operations ................................................ 39
  Section 4 – Survey Questions ............................................. 45
  Review of Sets .................................................................. 50

Unit 3 – Counting and Probability ........................................... 57
  Section 1 – Counting Methods .......................................... 57
  Section 2 – The Fundamental Counting Principle .............. 63
  Section 3 – Permutations and Combinations ...................... 66
  Counting Review ............................................................... 74

Unit 4 – Probability ............................................................. 78
  Section 1 – Basic Probability ............................................ 78
  Section 2 – Complements and Unions of Events .................. 84
  Section 3 – Conditional Probability and Intersections .......... 87
  Section 4 – Expected Value ............................................... 95
  Probability Review .......................................................... 98

Unit 5 – Statistics ............................................................... 103
  Section 1 – Organizing and Visualizing Data ...................... 103
Section 2 – Measures of Central Tendency .............................................. 107
Section 3 – Measures of Dispersion.......................................................... 111
Section 4 – The Normal Distribution.......................................................... 114
Section 5 – Linear Regression .................................................................... 120
Statistics Review .......................................................................................... 125
Unit 1 – Consumer Mathematics

Section 1 – Percent Change and Taxes

The word percent comes from the Latin “per centum” which translates to “per hundred.” Therefore, 37% means “thirty-seven per one hundred” which can also be written as $\frac{37}{100}$ or as 0.37.

Example 1: Convert the following to decimal form:

a. 18%

b. 68.02%

Example 2: Convert the following into percent form:

a. 0.046

b. 0.91
Example 3: Write the following in percent form:

a. \( \frac{5}{8} \)

b. \( \frac{11}{16} \)

Percent Change

The news media often use percentages to explain a change in some quantity. For example, you may hear the unemployment rate is down 4.1% or consumer confidence is up 11.2% over the last month. These statements use the notion of percent to compare changes in data over a period of time. This is commonly referred to as percent change.

Example 4: According to the Office of Management and Budget, in 2001 the U.S. government spent $305 billion for defense at a time when the federal budget was $1,860 billion. Ten years later in 2011, spending for defense was $706 billion and the budget was $3,602 billion. What percent of the federal budget was spent for defense in 2001? In 2011?
The percent change is always in reference to a previous or base amount. We then compare a new amount with the base amount as follows:

\[
\text{percent of change} = \frac{\text{new amount} - \text{base amount}}{\text{base amount}}
\]

If the new amount is less than the base amount, then the percent of change will be negative.

**Example 5:** According to the Office of Management and Budget, the defense budget for the U.S. was approximately $305 billion in 2001. In 2011, the defense budget had increased to $706 billion. Determine the percent of increase in the defense budget of the U.S. from 2001 to 2011.

**Example 6:** According to the U.S. Census, Arkansas had a population of 2,673,400 in 2000. After the U.S. Census in 2010, the population of Arkansas was reported as 2,915,916. Determine the percent of increase of Arkansas' population over these 10 years.
A *markup* is an increase that a merchant adds to the base price of an item. You can calculate *percent of markup* with the percent of change formula using the merchant’s cost as the base amount.

**Example 7:** Wolfe’s Auto is having a Labor Day sale in which the TV ads proclaim that all cars are sold at 2.99% markup over the dealer’s cost. In the ad, a new Toyota Tundra is on sale for $31,295. On the Internet, you find out that this particular model has a dealer cost of $24,339. Is the dealership being honest in his advertising?

---

**The Percent Equation**

All percent problems are variations of the same equation: \( \text{percent} \times \text{base} = \text{amount} \). This equation is referred to as the *percent equation*. In word problems it is often given in the following manner:

\[
\text{percent of a base is an amount}
\]

where the proposition “of” is used as an indicator of multiplication and the linking verb “is” represents an equality.

For example the statement “Eight is twenty-five percent of thirty-two” can be expressed as \( 8 = 0.25 \times 32 \) in equation form.
Example 8: Evaluate the following

a. What is 28% of 160?

b. 62 is 18% of what number.

c. 290 is what percent of 650?

d. A basketball team had a record of 51 wins and 31 losses. What percent of their games did they win?

e. In 2008, the average borrower graduated from a public college owing $21,250. This amount was up 117.125% from 1998. Determine the average amount of student loan debt that graduates from the school owed in 1998.
Example 9: The table below is taken from the instructions for filling out Form 1040 to compute federal income tax for a person whose marriage status is single. If Kasey is unmarried and has a taxable income of $36,358, what is the amount of federal income tax she owes?

<table>
<thead>
<tr>
<th></th>
<th>If your taxable income is Over—</th>
<th>But not Over—</th>
<th>The tax is</th>
<th>Of the amount Over—</th>
</tr>
</thead>
<tbody>
<tr>
<td>Line 1</td>
<td>$0</td>
<td>7,550</td>
<td>. . . . . . 10%</td>
<td>0$</td>
</tr>
<tr>
<td>Line 2</td>
<td>7,550</td>
<td>30,650</td>
<td>$755.00 + 15%</td>
<td>7,550</td>
</tr>
<tr>
<td>Line 3</td>
<td>30,650</td>
<td>74,200</td>
<td>$4,220.00 + 25%</td>
<td>30,650</td>
</tr>
<tr>
<td>Line 4</td>
<td>74,200</td>
<td>154,800</td>
<td>$15,107.50 + 28%</td>
<td>74,200</td>
</tr>
<tr>
<td>Line 5</td>
<td>154,800</td>
<td>336,550</td>
<td>$37,675.50 + 33%</td>
<td>154,800</td>
</tr>
<tr>
<td>Line 6</td>
<td>336,550</td>
<td>. . . . . .</td>
<td>$97,653.00 + 38%</td>
<td>336,550</td>
</tr>
</tbody>
</table>

Inflation is a rise in the level of prices of goods and services over a period of time. A common measure of inflation is the Consumer Price Index (CPI) which is maintained by the Bureau of Labor Statistics. As an example of inflation, the CPI Inflation Calculator provided on the Bureau of Labor Statistics site states that $2.00 in 1980 is the equivalent of $5.74 in 2014. As another example, the base CPI is 100 (which currently corresponds to average prices during the years 1982 to 1984). If you were to look up the CPI for dairy products in 2013, you would find it to be 218.0, which means that, on average, dairy products in 2013 cost 218% of what they cost in 1982 - 1984.

Example 10: If the average price for a carton of milk was $4.89 in 2013, how much, according to the CPI given above, would a carton of milk have cost, on average, in 1983?
Section 2 – Interest

Interest is the money that one person (the borrower) pays to another (a lender) to use the lender’s money. Borrowers pay interest; savers earn interest. In this section, we will explore two types of interest: Simple and Compound.

Simple Interest

Principal is the amount of money borrowed or deposited.

Interest rate is the rate at which the lender charges interest per one year (annually).

Simple Interest, then, is a calculation of how much interest is charged (or gained) for a given principal, rate, and time period.

\[ Simple \, Interest: \quad I = Prt \]

Example 1: If you deposit $800 in an account paying 3.2% annual interest, how much interest will the account earn in 4 years if the bank computes the interest using simple interest? How much money will be in the account after four years?
To determine the amount in an account at some time in the future, the *future value*, we add the interest to the principal. The principal is often called the *present value* of an account. Using $A$ to represent the future value, we can derive the formula for the *future value* of an account with simple interest:

$$ A = \text{principal} + \text{interest} $$

$$ A = P + I $$

$$ A = P + Prt $$

$$ A = P(1 + rt) $$

**Example 2:** If you deposit $1500 in an account paying 2.3% annual interest, how much will the account be worth at the end of six years?

**Example 3:** Assume you plan to save $3,000 for an epic road trip in two years. Your bank offers an account that pays 6% annual interest computed using simple interest. How much must you put into this account, now, in order to have the necessary money in 2 years?
**Compound Interest**

*Compound interest* is interest that is paid on principal *plus* previously earned interest. The time period allowed for interest to accrue before calculating compound interest is very important.

If interest is added yearly, it is compounded *annually*. If it is added every month, then interest is compounded *monthly*. *Quarterly* interest is interest computed every three months and *semi-annual* interest is interest computed every six months (or twice a year).

We can manipulate the simple interest formula to calculate the *future value* of an account that has interest compounded annually.

**Example 4:** $2,500 is deposited for three years in an account that pays 10% annual interest, compounded annually. How much will be in the account at the end of three years?

<table>
<thead>
<tr>
<th>Year</th>
<th>Principal (beginning of each year)</th>
<th>Future Value (end of year) $P(1 + rt)$</th>
</tr>
</thead>
<tbody>
<tr>
<td>1</td>
<td>$2,000</td>
<td></td>
</tr>
<tr>
<td>2</td>
<td></td>
<td></td>
</tr>
<tr>
<td>3</td>
<td></td>
<td></td>
</tr>
</tbody>
</table>
While we can continue to manipulate the simple interest formula to calculate compound interest, it is often simpler to use a \textit{compound interest} formula:

\[ A = P \left(1 + \frac{r}{m}\right)^{nt}, \quad n = mt \]

where \( P = \) principal, \( r = \) annual interest rate, \( m = \) number of times interest is compounded per year, \( n = \) the number of times interest is compounded for the situation and \( t = \) time (in years) for money to remain in the account.

\textbf{Example 5:} How much would be in an account at the end of thirty months if you deposited $2500 at an interest rate of 6.5\% compounded quarterly?

\textbf{Example 6:} A local store is running a promotion of “Six months same as cash” in which you will not have to make payments on store credit for six months for the purchase. If you do happen to pay off the purchase before six months is up, then you will not be charged interest. However if you do not pay off the purchase before six months, you are charged the full interest amount for the purchase for the six months. So, suppose you purchase a set of kitchen appliances at $3,500 under this promotion and do not make any payments for six months. Assuming the store is charging the standard store credit interest rate of 29.99\%, compounded monthly, how much interest do you owe at the end of the six months?
Example 7: Suppose upon the birth of a child, a parent wants to make a deposit into a tax-free account to use later for their child’s college education. Assuming this account has an annual interest rate of 4.9% compounded monthly, how much must the parent deposit now so the child will have $60,000 at age 18?

While there are rules of thumb, such as the rule of 70, to determine future values of investments, a graphing calculator can also be used to help determine the time period in which to leave an investment in order to receive a certain future value.

Example 8: Collectors often purchase classic videos hoping to get a good return on their investment. Jamie has invested $4600 in old Star Trek videos, which are currently increasing in value at 7.5% per year. If this increase continues at the same rate, in how many years will it take for Jamie’s investment to double? Triple?
The last situation we will consider is how to solve the compound interest equation $A = P \left(1 + \frac{r}{m}\right)^{mt}$ for $r$. To do this, we have to be able to solve equations such as $x^a = b$, where $a$ and $b$ are fixed numbers. To do this, we could raise both sides of the equation to the power of $\frac{1}{a}$ to remove the exponent from $x$. Alternately, we could use our graphing calculator, again.

**Example 9:** Vicki purchased a bond for $2400 to help establish a nature preserve. Two years later, she sold the bond for $4,620. What annual rate would she have to earn in a savings account compounded monthly to earn the same money on her investment?
Section 3 – Consumer Loans

Closed-Ended Credit

Consumer loans having a fixed number of payments are called closed – ended credit agreements (or installment loans). Each payment is called an installment and the interest charge is often called a finance charge. For closed-ended loans, figuring installment payments is somewhat straightforward.

To determine the monthly payment of an installment loan, add-on the interest to the principal and then divide by the number of months you have to pay the loan back to the lender. In other words,

\[
\text{monthly installment} = \frac{P + I}{n}
\]

where \(P\) is the amount of the loan, \(I\) is the total amount of interest due on the loan, and \(n\) is the number of monthly payments. Interest is computed with the simple interest formula, \(I = Prt\).

Example 1: If you take out a closed-ended loan for $1720 at an annual interest rate of 16% for three years, what will be your monthly payments?
Open-Ended Credit

With open-ended credit, you may be making monthly payments on a loan, but you may also be increasing the loan by making further purchases. An example of an open-ended credit account is a gas card or a store credit card. There are two methods for calculating finance charges for open-ended credit: the unpaid balance method and the average daily balance method.

The unpaid balance method computes interest based on the previous month's balance. This method uses the simple interest formula $I = Prt$, however

$$P = \text{previous month's balance}$$
$$+ \text{finance charge}$$
$$+ \text{purchase made}$$
$$- \text{returns}$$
$$- \text{payments}$$

and

$$r = \text{annual interest rate}, \ t = \frac{1}{12}.$$

Example 2: Assume the annual interest rate on your credit card is 21.99% and your unpaid balance at the beginning of last month was $800. Since then, you've purchased $150 in gas and sent in a payment of $100. Using the unpaid balance method, what is your credit card bill this month and what will be your finance charge for next month?
Example 3: Suppose that you graduate from college with $45,000 in student loans with a 6.99% interest rate. The company you borrowed from sets you up with monthly installments of $363. After your first payment, how much do you still owe on your loan?

Example 4: A credit card debt of $7,342 will be paid by making the minimum payment of $125. What will the balance be at the end of 1 month if the credit card company is using the unpaid balance method to compute finance charges and the annual interest rate on the card is 19.99%?
The *average daily balance method* is a common method used by credit card companies. With this method, the balance is the average of all daily balances for the previous month. In short, when you purchase something, you immediately begin to accumulate interest on the purchase. To compute this,

1. Add the outstanding balance for your account for each day of the month
2. Divide the total from step 1 by the number of days in the month to determine the average daily balance.
3. Use the average daily balance as the principal in \( I = Prt \) and set \( t = "\text{the number of days in the month divided by 365."} \)

**Example 5:** At the beginning of June, a credit card has a balance of $260. The card has an annual interest rate of 24.99%, and during June the following adjustments were made to the account:

- June 12 – payment of $50 was made
- June 19 – charge of $26 for gas
- June 22 – charge of $43 at a local store

Use the average daily balance method to compute the finance charge that will appear on the next statement.
Example 6: At the beginning of July, a credit card has a balance of $500. The annual interest rate is 21%. On July 5, a purchase of $300 was made and on July 17 a payment of $400 was credited. Calculate the finance charge that will appear on the statement for the next month using both the unpaid balance method and the average daily balance method.
Section 4 - Annuities

An annuity is an interest-bearing account into which a series of payments of the same size are made. If one payment is made at the end of every compounding period, the annuity is called an ordinary annuity. The future value of an annuity is the amount in the account, including interest, after making all payments. To calculate the future value of an annuity, we must calculate the interest earned on a series of payments. Consider the following example:

Example 1: Suppose that in January you begin making payments of $200 at the end of each month into an account paying 14% yearly interest compounded monthly. Determine the value of the annuity on July 1.
In general, the future value of an annuity can be calculated by

\[ A = R \left( \frac{1 + \frac{r}{m}}{\frac{r}{m}} \right)^n - 1 \]

where
- \( R \) = amount of regular deposit,
- \( r \) = annual percentage rate,
- \( m \) = number of times interest is compounded per year,
- \( n \) = the number of payments in the period.

**Example 2:** Assume that we make a payment of $60 at the end of each month into an account paying a 6.5% interest rate, compounded monthly. How much will be in the account at the end of three years?

If you pay into an annuity for a long enough period of time, the value of the annuity can become very large due to the compounding of interest. In fact, you can reach a time where the interest that the annuity earns exceeds the amount of deposits.

**Example 3:** Use your calculator to determine at what year an annuity in which you deposit $50 a month at 9.6% annual interest compounded monthly will have the interest exceed the total of the deposits.

\[ Y_1 = 50x \] (this is the equation for the amount in deposits)
\[ Y_2 = 6250(1.008^x - 1) \] (this is the simplified annuity formula)

Now, determine the intersection of the two graphs.
**Sinking Funds**

*A sinking fund* is an account you establish in which you save regularly until you have a fixed amount available. This special type of annuity does not have a new formula.

**Example 4:** Assume you wish to save $2400 in a sinking fund in two years. The account pays 6% compounded quarterly and you will also make quarterly payments. How much should you pay each quarter?

**Example 5:** Suppose you have decided to retire as soon as you have saved $1,000,000. Your plan is to put $300 each month into an ordinary annuity that pays a 5.5% interest rate compounded monthly. Use your graphing calculator to determine how many years it will be until you are able to retire.
Section 5 – Amortization (Optional)

The process of paying off a loan (plus interest) by making a series of regular, equal payments is called amortization, and such a loan is called an amortized loan. In order to determine your monthly payment, we will use two formulas from previous sections.

Assuming that you borrow a fixed amount, $P$, which you will repay by making payments on an amortized loan, you will make $m$ periodic payments per year for $n$ total payments ($n = mt$, $t$ in years) with an annual interest rate of $r$. Then you can find your payment by solving for $R$ in the equation

$$P \left(1 + \frac{r}{m}\right)^n = R \left(\frac{(1 + \frac{r}{m})^n - 1}{\frac{r}{m}}\right)$$

While the above equation is somewhat intimidating, it is important to note that the left-hand side is compound interest and the right hand side is an ordinary annuity. We have worked with both of these equations, already. As you will see, solving for $R$ will become a matter of simple division.

Example 1: An amortized loan of $21,000 is made to pay off a car in 5 years. If the yearly interest rate is 7.9%, what is your monthly payment?
Payments that a borrower makes on an amortized loan pay partly on principal and partly on interest on the outstanding principal. As the principal is reduced, each payment pays more and more toward principal. An amortization schedule is a list showing payment-by-payment how much is being paid toward principal and how much is paid toward interest.

Example 2: The Wooley family wishes to borrow $100,000 to finance a new house. They have obtained a 15 year mortgage at an annual rate of 6%, which has monthly payments of $1,198.20. Construct an amortization schedule for the first three payments of the loan.

<table>
<thead>
<tr>
<th>Payment Number</th>
<th>Monthly Payment</th>
<th>Interest Paid</th>
<th>Paid on Principal</th>
<th>Balance</th>
</tr>
</thead>
<tbody>
<tr>
<td>Month 0</td>
<td>-</td>
<td>-</td>
<td>-</td>
<td>$200,000</td>
</tr>
<tr>
<td>Month 1</td>
<td>1</td>
<td>$1,198.20</td>
<td></td>
<td></td>
</tr>
<tr>
<td>Month 2</td>
<td>2</td>
<td>$1,198.20</td>
<td></td>
<td></td>
</tr>
<tr>
<td>Month 3</td>
<td>3</td>
<td>$1,198.20</td>
<td></td>
<td></td>
</tr>
</tbody>
</table>
Present Value of an Annuity

If we know the monthly payment, the interest rate, and the number of payments, then the amount we can borrow is called the present value of the annuity. Using the formula below, we can determine the present value of an annuity by solving for \( P \).

\[
P \left( 1 + \frac{r}{m} \right)^n = R \left( \frac{\left( 1 + \frac{r}{m} \right)^n - 1}{\frac{r}{m}} \right)
\]

**Example 3:** If you can afford to spend $350 each month on car payments and the bank offers you a 4 year car loan with an annual rate of 12% and no money down, what is the present value of this annuity?

**Example 4:** Nikki has a retirement plan with an investment company. She can choose to be paid either $400 a month for 10 years or she can receive a lump sum of $30,000. Which is a better option? Assume an interest rate of 9%
Unpaid Balance of a Loan

To determine the unpaid balance of a loan,

\[ U = P \left( 1 + \frac{r}{m} \right)^n - R \left( \frac{\left( 1 + \frac{r}{m} \right)^n - 1}{\frac{r}{m}} \right) \]

**Example 4:** Suppose you have a 30 year mortgage for $125,000 at an annual interest rate of 9%. After 10 years, you refinance the loan to a lower rate. How much remains to be paid on your mortgage? If you refinance the remaining amount for 20 years at 6.1% interest, what are the new monthly payments? How much money have you saved in interest by refinancing?
1. Convert $\frac{3}{5}$ to a percent.

2. Convert $4\frac{7}{8}\%$ to a decimal.

3. 48% of 65 is what number?

4. 48 is what percent of 65?

5. 48% of what number is 65?

6. The average domestic crude oil price for a barrel of oil was $62.11 in 2006. The average price rose to $66.40 in 2007. What was the percent increase in the price from 2006 to 2007.
7. A silver necklace on sale for $39.99 was listed at 75% of the original price. What was the original price?

8. A local store advertised a savings of $80 on a tool set. What is the percent discount if the tool set was originally priced at $170.

9. Complete the first three lines of an amortization schedule for a $6000 loan with an annual interest rate of 12% over three years. (Question is from optional section 5.)

10. Anna bought a car for $14,000. To pay for it, she took a bank add-on interest loan at an annual interest rate of 4%. If the loan term is 8 years, what are her monthly payments?
11. What is the future value of a $6000 investment after five years, if the 9% interest rate is compounded monthly?

12. President Obama received 66,882,230 votes of the 125,225,901 votes cast for President during the 2008 election. What percent of the votes did he receive?

13. How much would you need to invest now if you wanted $1500 in three years at an interest rate of 4.25% compounded annually?

14. Based on the given information, use the unpaid balance method to find the finance charges on a credit card account. Last month’s balance $717, payment on account $259, interest rate 17%, purchases $118, returns $156.
15. Use the information from question 14 to determine the finance charges on a credit card account using the average daily balance method.

16. What is the present value of an account if the future value in two years will be $3720 at an interest rate of 10%, compounded annually?

17. Determine the monthly payment on a loan of $1860, if there was a $120 down payment and it was financed at a rate of 6% for 18 months.

18. Starting in January, you make monthly payments at the end of each month of $350 into an account with a yearly interest rate of 6% compounded monthly. How much money do you have available on November 1?
19. Anne purchased a bond from a museum valued at $15,000 for $9500. If the bond pays 6.5% annual interest compounded monthly, how long must she hold the bond until it reaches its full face value?

20. If you can afford to spend $250 each month on car payments and the bank offers you a 4 year car loan with an annual rate of 10% and no money down, what is the present value of this annuity?

21. Suppose you have a 30 year mortgage for $150,000 at an annual interest rate of 9.5%. After 10 years, you refinance the loan to a lower rate. How much remains to be paid on your mortgage? If you refinance the remaining amount for 20 years at 4.1% interest, what are the new monthly payments? How much money have you saved in interest by refinancing? (Question from optional section 5.)
**Unit 2 – Sets**

**Section 1 – The Language of Sets**

A collection of objects is called a *set*. A member of a set is an *element of the set*.

The *universal set* is the set of all elements under consideration in a given discussion. The universal set is often denoted by the capital letter $U$.

An *empty set* (also known as a *null set*) is a set without elements. The symbol for an empty set is $\emptyset$. An empty set can also be expressed by $\{ \}$. A set is *well-defined* if we are able to tell whether any particular object is an element of the set. In other words, determining if an element belongs to a particular set is not a matter of opinion.

Sets can be represented as a list of elements (also known as the *roster method*) or in *set – builder notation*. An example of each is below:

*Roster method:* $S = \{..., 2, 4, 6, 8, 10, ... \}$

*Set – builder:* $S = \{n | n \text{ is an even number} \}$

**Example 1:** Describe each of the following sets using the roster method.

a. $\{x | x \text{ is a natural number} \}$

b. $\{y | y \text{ is a number between 5 and 12, inclusive} \}$

c. $\{z | z \text{ is a letter in the word } university \}$
Example 2: Describe each of the following sets using set-builder notation.

a. \{3, 6, 9, 12, 15\}

b. \{0, 1, 8, 27, 64, 125, ...\}

c. \{Asia, Europe, Africa, North America, South America, Australia, Antarctica\}

Example 3: Determine if the given set is well-defined.

a. \{x | x is a current professor at UAM\}

b. \{y | y is the name of a famous actor\}

c. \{z | z is the name of a movie that has the word "Trek" in its title\}

d. \{w | w is an unlucky day\}

Infinite and Finite Sets

The cardinal number of a set is the number of elements in a finite set. A finite set is a set in which the elements of the set are capable of being counted. A shorthand notation for cardinal number is \(n(S)\) in which \(n\) reminds us of the word number and \(S\) is the name of the set under consideration.

The set \(A = \{3, 6, 9, 12, 15\}\) has a cardinal number of \(n(A) = 5\).
An empty set has a cardinal number of 0.

The set \( \{0, 1, 8, 27, 64, 125, \ldots\} \) does not have a defined cardinal number since it is an infinite set. An infinite set is a set in which the elements cannot be counted.

**Example 4:** Classify the following sets as infinite or finite.

a. \( \{1, 2, 3, 4, 5, 6, \ldots\} \)

b. \( \{1, 2, 3, 4, 5, 6, \ldots, 100\} \)

c. \( \{x| x \text{ is a person who has received a parking ticket}\} \)

d. \( \{y| y \text{ is a whole number such that } y < 0\} \)

e. \( \{z| z \text{ is a rational number such that } 1 < z < 2\} \)

**Example 5:** What is the cardinal number of the following sets?

a. \( N = \{x| x \text{ is a natural number}\} \)

b. \( K = \{y| y \text{ is a number between 5 and 12, inclusive}\} \)

c. \( S = \{z| z \text{ is a letter in the word university}\} \)

d. \( T = \{w| w \text{ is a whole number less than 0}\} \)

e. \( X = \{\{1, 4, 8\}, \{5, 9, 0\}, \{2\}\} \)
Section 2 – Comparing Sets

Two sets, \( A \) and \( B \), are equal if they have exactly the same members. In this case, we write \( A = B \). If \( A \) and \( B \) are not equal, then \( A \neq B \).

**Example 1:** Determine which of the following sets are equal and explain your answer.

\[
A = \{x \mid x^2 - 9 = 0\} \\
B = \{y \mid \text{the absolute value of } y \text{ is } 3\} \\
C = \{z \mid z \text{ is an integer that lies } 3 \text{ units of less from } 0\}
\]

Equivalent sets are sets in which the sets have the same number of elements; that is, for sets \( A \) and \( B \), \( A \) is equivalent to \( B \) if \( n(A) = n(B) \).

**Example 2:** Determine which of the following sets are equivalent and explain your answer.

\[
D = \{p, i, g\} \\
E = \{c, o, w\} \\
F = \{f \mid f \text{ is a factor of } 12\} \\
H = \{h \mid h \text{ is a letter in the word } \text{“book”}\}
\]
It is important to note the difference between *equal* and *equivalent* sets. All *equal sets* are also *equivalent*. However, *equivalent sets* may not be *equal*.

Two more definitions to be aware of are the definitions of *subset* and *proper subset*.

The set $A$ is a *subset* of the set $B$ if every element of $A$ is also an element of $B$. This relationship is indicated by $A \subseteq B$. If $A$ is not a subset of $B$, we write that as $A \not\subseteq B$. As an example, $A = \{1, 2, 3\}$ is a subset of $B = \{-1, 0, 1, 2, 3, 4\}$. The *empty set* is a subset of every set.

A *proper subset* is a subset in which $A \subseteq B$, but $A \neq B$. In the example in the line above, set $A$ is a proper subset of set $B$. To indicate this, we would write $A \subset B$. Notice this symbol does not have the line under it. This indicates that the two sets are not equal.

If we wanted to show that two sets were not *proper subsets* we would write $C \not\subset D$.

**Example 3:** Is the set of rational numbers a subset of real numbers? Is the set of rational numbers a proper subset of real numbers?

**Example 4:** Is the set of rational numbers a subset of irrational numbers? Is the set of rational numbers a proper subset of irrational numbers?

**Example 5:** Let $A = \{Steven Colbert, Jon Stewart, Jimmy Fallon\}$ and let $B = \{x | x \text{ is a comedian}\}$. Is set $A$ a subset of set $B$? Is set $A$ a proper subset of set $B$? Is $B$ a subset of $A$?
To list all the subsets of a given set, it is often a good idea to write the subsets from smallest to largest in terms of cardinal numbers. For example,

If \( C = \{1, 2\} \), the subsets of \( C \) are

<table>
<thead>
<tr>
<th>Subset</th>
<th>Size (Cardinal Number)</th>
</tr>
</thead>
<tbody>
<tr>
<td>( \emptyset )</td>
<td>0</td>
</tr>
<tr>
<td>( {1} )</td>
<td>1</td>
</tr>
<tr>
<td>( {2} )</td>
<td>1</td>
</tr>
<tr>
<td>( {1,2} )</td>
<td>2</td>
</tr>
</tbody>
</table>

**Example 6:** For the following sets, determine their cardinal number, list their subsets, and determine the total number of subsets and proper subsets.

<table>
<thead>
<tr>
<th>Set ( A )</th>
<th>( k = n(A) )</th>
<th>Subsets of set ( A )</th>
<th>Number of Subsets</th>
<th>Number of Proper Subsets</th>
</tr>
</thead>
<tbody>
<tr>
<td>( \emptyset )</td>
<td></td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td>( {x} )</td>
<td></td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td>( {x, y} )</td>
<td></td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td>( {x, y, z} )</td>
<td></td>
<td></td>
<td></td>
<td></td>
</tr>
</tbody>
</table>

a. The relationship between \( k = n(A) \) and the number of subsets is

b. The relationship between \( k = n(A) \) and the number of proper subsets

c. If set \( A \) had 8 elements, then how many subsets and proper subsets would it have?
Section 3 – Set Operations

A Venn diagram is a graphic organizer than can be used to illustrate subset and set relations. To draw a Venn diagram, you first draw a rectangle to represent your universal set. Then, draw circles inside the rectangle to represent the subsets of the universal set. If the subsets overlap on elements, show the circle overlapping in the Venn diagram.

For example, if you have the universal set $U = \{-1, 0, 1, 2, 3, 4, 5, 6\}$ and want to show how the subsets $A = \{0, 1, 2, 3\}$ and $B = \{2, 3, 4, 5\}$ relate to each other, your Venn diagram would look like this:

Where the two subsets overlap is referred to as an intersection of sets $A$ and $B$ and is the set of elements that are common to both $A$ and $B$. The symbol for the relationship is $\cap$, and we can now write $A \cap B = \{2, 3\}$. Sets that do not have any common elements are disjoint.

Another relationship between sets is the relationship of a union. The union of sets $A$ and $B$, written $A \cup B$, is the set of elements that are elements of either $A$ or $B$, or both. We can write for the example above, $A \cup B = \{0, 1, 2, 3, 4, 5\}$.

The complement of $A$, written $A'$, is the set of elements in the universal set that are not elements of $A$. So, $A' = \{-1, 4, 5, 6\}$.

Finally, the difference of sets $B$ and $A$, written $B - A$, is the set of elements in $B$ but not in $A$. For the above example, $B - A = \{4, 5\}$ and $A - B = \{0, 1\}$.
Example 1: Let $U = \{t, r, a, m, p, o, l, i, n, e\}$, $A = \{l, i, n, e\}$, $B = \{p, o, l, i, t, e\}$, $C = \{r, a, m, p\}$ and $D = \{p, i, n, e\}$

a. $A \cap B$

b. $A \cup B$

c. $A'$

d. $B'$

e. $A' \cap B'$

f. $(A \cup B)'$

g. Compare sets $e$ and $f$, what do you observe?

h. $B \cup D$

i. $C \cap (B \cup D)$

j. $C \cap B$

k. $C \cap D$

l. $(C \cap B) \cup (C \cap D)$

m. Compare sets $i$ and $l$. What do you observe?

n. $A - B$

o. $B - A$

p. $A \cup \emptyset$

q. $A \cap \emptyset$
Example 2: Shade the following Venn diagrams as indicated:

![Venn diagrams](image)

What do you notice about the last pair of Venn Diagrams?
Example 3: Shade the following Venn diagrams as indicated.

What do you notice about the last pair of Venn diagrams?
Order of Set Operations

Just as we must follow the order of operations when we work algebra problems, set notation specifies the order in which we perform set operations. The previous examples illustrated the importance of order of set operations. Additionally, three laws were illustrated many times in the previous examples.

DeMorgan’s Laws for Set Theory:

If $A$ and $B$ are sets, then $(A \cup B)' = A' \cap B'$ and $(A \cap B)' = A' \cup B'$.

Distributive Property

$A \cap (B \cup C) = (A \cap B) \cup (A \cap C)$

Inclusion – Exclusion Principle

$n(A \cup B) = n(A) + n(B) - n(A \cap B)$

Example 4: Let $A = \{r, a, i, n\}$, $B = \{w, a, r, n\}$, and $C = \{b, o, w\}$

a. $n(A) =$

b. $n(B) =$

c. $n(A \cup B) =$

d. $n(C) =$
e. $n(C \cup B) =$
**Example 5:** Let $U = \{x \mid x \text{ is a natural number less than } 12\}$, $A = \{y \mid y \text{ is a factor of } 9\}$, and $B = \{z \mid z \text{ is a factor of } 12\}$

a. $A \cup B = \ldots$

b. $n(A \cap B) = \ldots$

c. $n(A) + n(B) - n(A \cap B)$

d. $n(A \cup B) = \ldots$

e. $A - B = \ldots$

f. $B' = \ldots$

g. $A' = \ldots$

h. $A' \cap B' = \ldots$

i. $(A \cup B)' = \ldots$

**Example 6:** Name the shaded region in each of the following Venn diagram.

![Venn Diagrams](images/venn_diagrams.png)
Section 4 – Survey Questions

Survey questions are a natural part of data collection for analysis. Often, the data collected is displayed in Venn diagrams in order to have a better understanding of how the responses of the individuals participating in the survey relate to each other.

Before we work a few survey questions, let’s review:

Example 1: Name the regions in the following:

a. \( A \cup B = \) ____________

b. \( A \cap B = \) ____________

c. \( A' = \) ____________

d. \( B - A = \) ____________

Example 2: Name the regions in the following:

a. \( A \cap (B \cup C) = \) ____________

b. \( (A \cap B) - C = \) ____________

c. \( (A \cup C)' = \) ____________

d. \( (A \cap B)' = \) ____________
**Example 3:** Use the following information to fill in the Venn diagram and determine \( n(B) \) and \( n(C) \).

\[
\begin{align*}
n(A) &= 22, n(A \cap B \cap C) = 5, n(A \cap B) = 9 \\
n(B - A) &= 14, n(C - A) = 19, n(B \cup C) = 32 \\
\text{and } n(B \cap C) &= 16.
\end{align*}
\]

\( n(B) = \_ \_ \_ \_ \_ \_ \_ \_ \)  

\( n(C) = \_ \_ \_ \_ \_ \_ \_ \_ \)

The example above is actually a survey question stripped of its relevant data. The next few examples will be survey questions with the relevant data included.

**Example 4:** The campus newspaper is interested in common factors in automobile accidents involving members of the student body. After sending out a questionnaire through email, these are the responses gathered from the student body:

- 18 recent accidents involved the driver texting and excessive speed
- 26 recent accidents involved the driver texting
- 12 recent accidents involved excessive speed but not the driver texting
- 21 recent accidents involved neither excessive nor the driver texting

How many of the accidents...

a) Involved texting only?  
b) Involved speeding but not texting?  
c) Were reported to the campus newspaper?
Example 5: A survey of 90 people who called into a radio talk show about sports were asked which of the three types of sports they enjoyed. Listed below are the results.

- 5 people liked baseball, basketball, and football
- 15 people liked baseball and basketball
- 25 people liked football and basketball
- 10 people liked baseball and football
- 25 people liked baseball
- 40 people liked football
- 55 people like basketball

Fill in the Venn diagram and answer the questions that follow.

a. How many liked at least one of these types of sports? ___________

b. How many liked baseball or football? ___________

c. How many liked baseball or football but not basketball? ___________

d. How many liked baseball and football but not basketball? ___________

e. How many liked only one of these three types of sports? ________

f. How many liked exactly two of these three types of sports? ________

g. How many did not like any of the three types of sports? ___________

h. How many liked baseball only? ___________

i. How many liked football only? ___________
Example 6: The Provost of the university surveyed a group of students about which support services they were using to help improve their academic performance and found the following results:

5 were using office hours, tutoring, and study groups
16 were using office hours and tutoring
28 were using tutoring
14 were using tutoring and study groups
8 were using office hours and study groups but not tutoring
23 were using office hours but not tutoring
18 were using only study groups
37 were using none of these services

Fill in the Venn diagram below and answer the following questions.

a) How many students were surveyed?

b) How many students were using only office hours?

c) How many students were using study groups?

d) How many students used at least two of the support services?

e) How many students used tutoring but not office hours?

f) How many students used office hours only?
Example 7: A survey was taken in which 100 students were asked which of the three ways they used to communicate with friends and family. Their responses are recorded below:

17 used Texting, Facebook, and Twitter
28 used Facebook and Twitter
24 used Twitter and texting
42 used texting but not Facebook
86 used texting or Twitter
14 used only Twitter
14 used none of these three methods

Fill in the Venn diagram above and then answer the following questions:

a) How many use Facebook or texting?

b) How many used Facebook or texting but not Twitter?

c) How many used Facebook and texting but not Twitter?

d) How many used Facebook only?

e) How many used exactly one of these ways to communicate?

f) How many used at least one of these ways to communicate?

g) How many used at least two of these ways to communicate?
**Review of Sets**

For problems 1-8, consider the following sets: 

\[ U = \{g, r, e, a, t, n, s\}, \]

\[ A = \{g, a, s\}, \]

\[ B = \{r, e, s, t\}, \]

\[ C = \{e, a, t\}. \]

1. \[ A \cup B = \]

2. \[ A \cap C = \]

3. \[ A - B = \]

4. \[ A^{\prime} = \]

5. \[ C \cup (A \cap B) = \]

6. The number of subsets of B is

7. The number of proper subsets of A is

8. \[ n(U) = \]
9. \((A \cup B')' =\)
   \(A)\) \(A' \cap B\) \(B)\) \(A' \cup B\) \(C)\) \(A \cap B'\) \(D)\) \(A' \cap B\)

10. \(A \cup (B \cap C) =\)
    \(A)\) \((A \cap B) \cup (A \cap C)\) \(D)\) \((A \cup B) \cap (A \cup C)\)
    \(B)\) \((A \cap B) \cap (A \cap C)\) \(E)\) \((A \cup B) \cap (A \cup C)\)
    \(C)\) \((A \cup B) \cup (A \cup C)\)

11. Write the set by listing the elements. \(\{a | a\ is\ an\ integer\ greater\ than\ 7\}\)

12. Write the set using set-builder notation. \(\{0, 1, 4, 9, 16, 25, 36\}\)

13. Which of the following sets is finite?
   \(A)\) \(\{a : a\ is\ a\ real\ number\ between\ 2\ and\ 3\}\)
   \(B)\) \(\{b : b\ is\ an\ integer\ less\ than\ 0\}\)
   \(C)\) \(\{c : c\ is\ a\ prime\ number\}\)
   \(D)\) \(\{d : d\ is\ a\ student\ enrolled\ at\ UAM\ for\ the\ Spring\ 2013\ semester\}\)
   \(E)\) \(\{e : e\ is\ a\ multiple\ of\ 2\}\)
14. If \( E = \{1, 3, 9\} \), then \( E \) could also be expressed as which of the following?

A) \( E = \{e : e \text{ is a positive integer factor of 9}\} \)

B) \( E = \{e : e \text{ is a prime number less than 10}\} \)

C) \( E = \{e : e \text{ is a positive multiple of 9}\} \)

D) \( E = \{e : e \text{ is an odd number less than 10}\} \)

E) \( E = \{e : e \text{ is a positive integer factor of 18}\} \)

15. Which of the following sets is well-defined?

A) \( \{d : d \text{ is a challenging math problem}\} \)

B) \( \{m : m \text{ is an easy yoga move}\} \)

C) \( \{c : c \text{ is an animal that makes a great pet}\} \)

D) \( \{y : y \text{ is an elderly person}\} \)

E) \( \{v : v \text{ is a UAM student who has a GPA of over 3.25}\} \)

16. In which of the following sets would you find Beyoncé Knowles?

A) \( A \cap B \)

B) \( B - A \)

C) \( A - B \)

D) \( A' \)

E) \( A' \cap B \)

17. In which of the following sets would you find Garth Brooks?

A) \( A \cap B \)

B) \( B - A \)

C) \( A - B \)

D) \( A' \)

E) \( A' \cap B \)
18. Which of the following sets is equivalent to \( A = \{ a : |a| = 3 \} \)?

A) \( A = \{ a : \text{a day of the week that starts with the letter W} \} \)
B) \( A = \{ a : a \text{ is a day of the week that starts with the letter T} \} \)
C) \( A = \{ a : a \text{ is the name of a state that starts with the letter A} \} \)
D) \( A = \{ a : a \text{ is the name of a former U.S. president} \} \)
E) \( A = \{ a : a \text{ is a prime number less than 6} \} \)

19. What region is shaded in the following Venn diagram?

A) \( A' \)  
B) \( B' \)  
C) \( A - B \)  
D) \( B - A \)  
E) \( (A \cap B)' \)

20. What region is shaded in the following Venn diagram?

A) \( (A \cup B)' \)  
B) \( (A \cap B)' \)  
C) \( (A \cup B)' \cap C \)  
D) \( A' \cup B' \cup C \)  
E) \( A' \cap B' \cap C \)

21. Determine the cardinal number of \( X = \{ a, \{ b \}, \{ \emptyset \} \} \).
22. Fill in each region in the following Venn diagram with a number based on the clues given below. Use these numbers to find \( n(A) \). There is no number outside the circles.

\[
\begin{align*}
\text{n}(A \cap B) &= 7, \\
\text{n}(A \cap B \cap C) &= 3, \\
\text{n}(B \cap C) &= 9, \\
\text{n}(B - A) &= 12, \\
\text{n}(B \cup C) &= 23, \\
\text{n}(A \cap C) &= 5, \\
\text{n}(A \cup B \cup C) &= 28.
\end{align*}
\]

\[\text{n}(A) \phantom{\quad} \phantom{\quad} \phantom{\quad} \phantom{\quad} \phantom{\quad} \phantom{\quad} \phantom{\quad} \phantom{\quad} \phantom{\quad} \phantom{\quad} \phantom{\quad} \phantom{\quad} \phantom{\quad} \phantom{\quad} \phantom{\quad} \phantom{\quad} \phantom{\quad} \phantom{\quad} \phantom{\quad} \phantom{\quad} \phantom{\quad} \phantom{\quad} \phantom{\quad} \phantom{\quad} \phantom{\quad} \phantom{\quad} \phantom{\quad} \phantom{\quad} \phantom{\quad} \phantom{\quad} \phantom{\quad} \phantom{\quad} \phantom{\quad} \phantom{\quad} \phantom{\quad} \phantom{\quad} \phantom{\quad} \phantom{\quad} \phantom{\quad} \phantom{\quad} \phantom{\quad} \phantom{\quad} \phantom{\quad} \phantom{\quad} \phantom{\quad} \phantom{\quad} \phantom{\quad} \phantom{\quad} \phantom{\quad} \phantom{\quad} \phantom{\quad} \phantom{\quad} \phantom{\quad} \phantom{\quad} \phantom{\quad} \phantom{\quad} \phantom{\quad} \phantom{\quad} \phantom{\quad} \phantom{\quad} \phantom{\quad} 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24. The number of male and female students are shown by semester in the table below. Using the letters in the table, find the cardinal number of each set. (6 points)

<table>
<thead>
<tr>
<th>Survey of Math Class</th>
<th>Male (M)</th>
<th>Female (F)</th>
<th>Total</th>
</tr>
</thead>
<tbody>
<tr>
<td>Fall 2012 (A)</td>
<td>7</td>
<td>18</td>
<td>25</td>
</tr>
<tr>
<td>Spring 2013 (B)</td>
<td>6</td>
<td>23</td>
<td>29</td>
</tr>
<tr>
<td><strong>Total</strong></td>
<td><strong>13</strong></td>
<td><strong>41</strong></td>
<td><strong>54</strong></td>
</tr>
</tbody>
</table>

\[ n(B \cup F) = \quad \quad n(A \cap M) \quad \quad \]

25. A survey was conducted in which a randomly selected group of UAM students was asked the question, “Which of the following three movies have you seen: Les Miserables, Lincoln and Silver Linings Playbook?” The responses are summarized below.

19 said they had seen Lincoln
13 said they had both Lincoln and Les Miserables
5 said they had seen Les Miserables and Silver Linings Playbook
12 said they had seen none of the three movies
2 said they had seen all three movies
6 said they had seen Lincoln and Silver Linings Playbook
20 said they had seen Les Miserables
10 said they had seen Silver Linings Playbook

*Continued on next page*
Fill in the following Venn diagram and answer the questions below.

How many students were involved in this survey? ______

How many said that they only saw Lincoln? ______

How many said they saw Les Miserables or Lincoln? ______

How many said they saw Les Miserables or Lincoln but not Silver Linings Playbook? ___

How many said they saw at most two of the three movies? ______

How many said they saw at least two of the three movies? ______
In this section, you will begin to learn several techniques for counting large sets of items systematically and quickly. One of the simplest ways to count is by systematic listing.

**Example 1:** Lynn’s friends Andrea, Barbie, Carrie, and Denise have volunteered to help with the preparations for a party. How many ways can Lynn assign someone to buy beverages, someone to arrange for food, and someone to send out invitations? Assume that no person does two jobs. List the ways the jobs could be assigned.
A tree diagram is another method in which you count arrangements of options. The concept of the tree diagram is based upon the idea of events happening in sequence. The total number of options is determined by the number of branches at the end of the diagram.
Example 2: Suppose you are buying a flavor combination drink at a local restaurant. There are four flavors you can add to your soda: vanilla, lemon, chocolate, and strawberry. The flavors can be repeated or not and two drinks are considered to be different if the flavors are the same but occur in different orders. How many different drinks are possible that have all four flavor additions? Draw a tree diagram to illustrate the different ways to add the flavors to your soda.
Example 3: Suppose that you are taking a four question true/false quiz. Draw a tree diagram to show all the possible ways that you could answer the questions on the quiz. Let $C$ represent the question was answered correctly and $I$ represent the question was answered incorrectly.

How many ways can you answer all the questions incorrectly?

How many ways can you answer exactly three questions incorrectly?

How many ways can you answer exactly two questions incorrectly?
Example 4: How many three digit numbers can be formed from the set of digits \{0,1,2,3\}? Note: 0 cannot be used as the hundred’s digit and the digits may be repeated.

Example 5: How many three digit numbers can be formed from the set of digits \{0,1,2,3\}? Note: 0 cannot be used as the hundred’s digit and the digits may NOT be repeated.
There are many situations in which tree diagrams can help us visualize the options available for counting. Unfortunately, many of those situations involve hundreds or more of branches at the end of the diagram. In such cases, extending the branches along one or two paths is often enough to determine the total number of options in the situation.

**Example 6:** How many 3 digit numbers can you form from the digits \{0, 1, 2, 3, 4\} if repetitions are allowed and zero cannot be in the hundred’s digit?

**Example 7:** How many 3 digit numbers can you form from the digits \{0, 1, 2, 3, 4\} if repetitions are not allowed and zero cannot be in the hundred’s digit?

**Example 8:** How many 3 digit numbers can you form from the digits \{0, 1, 2, 3, 4\} if repetitions are not allowed, the 3 digit numbers must be even, and zero cannot be in the hundred’s digit?
Section 2 – The Fundamental Counting Principle

The fundamental counting principal (FCP) solves problems without listing elements or drawing tree diagrams. With the FCP, you have a basic pattern:

A first event happens in \( a \) number of ways,
then a second even happens in \( b \) number of ways,
then a third event happens in \( c \) number of ways,
then....until there are no more events left.

As a result, all the events can happen in
\( (a)(b)(c) \) ... ways.

Example 1: If we have 5 ab machines, 4 arm machines, and 8 cardio machines, how many different workouts can we have if we use one machine from each group? Work this example first with a tree diagram, then with the FCP.

Tree Diagram:
The blanks used in the FCP portion of example 1 are referred to as a *slot diagram*. Drawing this type of diagram is a useful technique for keeping track of the number of ways to perform each task.

**Example 2:** How many ways can four coins be flipped?

\[
\begin{array}{cccc}
\bl & \times & \bl & \times & \bl & \times & \bl \\
1^{st} \text{ coin} & & 2^{nd} \text{ coin} & & 3^{rd} \text{ coin} & & 4^{th} \text{ coin}
\end{array}
\]

**Example 3:** How many ways can three dice (red, blue, green) be rolled?

\[
\begin{array}{ccc}
\bl & \times & \bl \\
1^{st} \text{ die} & & 2^{nd} \text{ die} & & 3^{rd} \text{ die}
\end{array}
\]

**Example 4:** If David has three sport coats, five pairs of pants, seven shirts, and four ties, how many different ways can he select an outfit consisting of a coat, pants, shirt, and tie?

\[
\begin{array}{cccc}
\bl & \times & \bl & \times & \bl & \times & \bl \\
\text{Coat} & & \text{Pants} & & \text{Shirt} & & \text{Tie}
\end{array}
\]

**Example 5:** Suppose that you want to construct a six-character password for one of your online accounts. How many passwords are possible subject to the following conditions?

a) The password can consist of letters and digits in any order and is not case sensitive. Symbols may not be repeated.

b) The password consists of letters and digits in any order and is case sensitive. Symbols may be repeated.
Example 6: In a certain state, license plates currently consist of two letters followed by three digits. How many such license plates are possible? If the state department of transportation decides to change the plates to have three letters followed by two digits, how many plates will now be possible?

Example 7: How many different seating arrangements are possible in the following situation? Assume there are 12 students in a class and the front row has six chairs. Also, the professor wants Andy to sit between his tutor and Brad. It does not matter on which side of Andy the tutor sits. Andy, Brad, and the tutor are students in the class. How many ways can the professor arrange the front row?
Section 3 – Permutations and Combinations

Permutations

A permutation is an ordering of distinct objects (they cannot be repeated) in a straight line. If we select \( r \) different objects from a set of \( n \) objects and arrange them in a straight line, this is called a permutation of \( n \) objects taken \( r \) at a time. The number of permutations of \( n \) object taken \( r \) at a time is denoted by \( P(n, r) \) or \( n \, P \, r \).

In a permutation, order matters. In other words, \( AB \neq BA \).

Example 1: How many permutations are there of \( a, b, c, \) and \( d \)? Write the answer using \( P(n, r) \) notation.

\[
\begin{array}{cccc}
1^{\text{st}} \text{letter} & 2^{\text{nd}} \text{letter} & 3^{\text{rd}} \text{letter} & 4^{\text{th}} \text{letter} \\
\quad & \times & \quad & \end{array}
\]

Any letter \quad Can't repeat \quad Can't repeat \quad Can't repeat

\[\begin{array}{cccc}
\end{array} = \]

\[ P(n, r) = P( \quad ) = \text{___________} \]

Example 2: How many permutations are there of the letters \( p, q, r, s, t, u, \) and \( v \) if we take the letters three at a time? Write the answer using \( P(n, r) \) notation.

\[
\begin{array}{ccc}
1^{\text{st}} \text{letter} & 2^{\text{nd}} \text{letter} & 3^{\text{rd}} \text{letter} \\
\quad & \times & \quad \\
\end{array}
\]

Any letter \quad Can't \quad Can't repeat \quad repeat

\[ \begin{array}{ccc}
\quad & \times & \quad \\
\end{array} = \]

\[ P(n, r) = P( \quad ) = \text{___________} \]
Before the formula to compute $P(n, r)$ can be given, the concept of *factorial* must be defined. A factorial is a mathematical expression in which a product is taken of decreasing integer values until the values reach 1. An example of a factorial would be $8! = 8 \times 7 \times 6 \times 5 \times 4 \times 3 \times 2 \times 1$. Below is a more formal definition of *factorial*:

If $n$ is a counting number, the symbol $n!$, called $n$ *factorial*, stands for the product $n \times (n - 1) \times (n - 2) \times (n - 3) \times \ldots \times 2 \times 1$. We define $0! = 1$.

**Example 3:** Compute the following:

a. $5!$

b. $(9 - 2)!$

c. $\frac{7!}{4!}$

d. $\frac{10!}{7!3!}$

The formula for computing $P(n, r) = \frac{n!}{(n-r)!}$. 
Example 4: A committee of 12 people decide to select, from their committee members, one person to serve as president, a second to serve as vice president, and a third to serve as secretary. In how many ways can the committee fill these positions?

Example 5: On a science quiz, a student must match 10 terms with their definitions. If the same term cannot be used twice, how many ways can a student match the terms to the definitions?

Example 6: To open your locker in the break room at work, you must enter five digits in order from the set 0, 1, 2, 3, 4, 5, 6, 7, 8, 9 into a keypad and you cannot repeat a digit. How many different keypad pattern are possible for your locker?
**Combinations**

If we slightly change the conditions in the three previous examples, the examples become a different type of counting method. For example,

**Example 7:** A committee of 12 people decide to select, from their committee members, a three person subcommittee to work together in creating long term goals. In how many ways can the committee fill these positions?

To see how this is slightly different from example 4, create a tree diagram to help visualize the number of options available.

The situation in example 7 is referred to as a *combination*. If we choose $r$ objects from a set of $n$ objects and ORDER DOES NOT MATTER, we say that we are forming a *combination* of $n$ objects taken $r$ at a time. The notation $C(n, r)$ or $nCr$ denotes the number of such combinations.

The formula for $C(n, r) = \frac{P(n, r)}{r!} = \frac{n!}{r!(n-r)!}$.
Example 8: Answer the following:

a. How many four-element sets can be chosen from a set of six objects?

b. How many four-person committees can be formed from a group of eleven people?

It is perhaps interesting to note that $C(n, r)$ is usually smaller than $P(n, r)$. One reason for this is that in a combination, order does not matter, so $AB = BA$ and the combination of $A$ and $B$ is only counted once. For a permutation, recall that $AB \neq BA$ so the combining of $A$ and $B$ is counted twice.

Example 9: In poker, five cards are drawn from a standard 52-card deck. How many different poker hands are possible?

Example 10: Five players are to be chosen from a 25-player Major League Baseball team to visit schools to support increasing exercise during summer vacation. In how many ways can this selection be made?
Combinations and Pascal’s Triangle

Pascal’s triangle is a geometric arrangement of rows of numbers that create a certain numerical pattern. The first few rows of Pascal’s triangle is given below. See if you can spot the pattern.

\[
\begin{array}{cccc}
1 &  &  & \\
1 & 1 &  & \\
1 & 2 & 1 & \\
1 & 3 & 3 & 1 \\
1 & 4 & 6 & 4 & 1 \\
\end{array}
\]

The first row of the triangle is labeled Row 0 and the leftmost position on each row is given the position of zero. There are many reasons for why we number the rows and positions in the triangle as we do and one of those reasons relates to combinations. If rewrite the entries of Pascal’s triangles as combinations, this is what we have:

\[
\begin{align*}
C(0,0) = 1 \\
C(1,0) = 1, \quad C(1,1) = 1 \\
C(2,0) = 1, \quad C(2,1) = 3, \quad C(2,2) = 1 \\
C(3,0) = 1, \quad C(3,1) = 4, \quad C(3,2) = 6, \quad C(3,3) = 1
\end{align*}
\]
So, in Pascal’s triangle, \( C(17, 5) \) is the number on row 17 (physically, 18 rows from top) and position 5 (six numbers from the left, physically) in the row.

**Example 11:** What is the 8th number from the left in Row 21 of Pascal’s triangle?

Finally, Pascal’s triangle can help us count the different combinations of number in subsets available for each set.

So, for a set with \( k \) elements, there are \_________ subsets
and
\_________ proper Subsets.

On Row 0, the total number of subsets is ________.
On Row 1, the total number of subsets is ________.
On Row 2, the total number of subsets is ________.
On Row 3, the total number of subsets is ________.
On Row 4, the total number of subsets is ________.
Also, the \((row, position\ in\ row)\) arrangement tells us the number of subsets of a particular size for a set with \(k = row\ number\) elements.

For example, the number in Row 5, position 2 of Pascal’s triangle is 10. This entry tells us that for a set of \(k = 5\) elements, there are 10 subsets with 2 elements each.

**Example 12:** Answer the following questions:

a. If the cardinal number of a set is 16, how many three element subsets would the set have?

b. If the cardinal number of a set is 21, how many 12 element subsets does the set have?

**Combining Counting Methods**

There are times when a situation requires the use of the FCP, permutations, and combinations. Once again, it is helpful to use a slot diagram to organize the numbers.

**Example 14:** Suppose you have lunch at Mandy’s Diner and decide to purchase her combo platter. The combo platter consists of a salad, two entries, and three vegetables. How many different combo platters are available if you have the choice of 6 salads, 8 entrees, and 12 vegetables?

**Example 15:** The student government organization is selecting three women and two men to attend a leadership conference. If 12 men and 14 women are qualified for the conference, how many different ways can the organization make the decision?


**Counting Review**

1. Assume you are rolling two dice, the first one is red and the second one is green. How many ways can you roll a sum of 9 on the two dice?

2. If you draw a tree diagram showing how many ways 5 coins could be flipped, how many branches would it have?

3. The owner of a stereo store wants to advertise that she has many different sound systems in stock. The store carries 10 different CD players, 8 different receivers, and 7 different speakers. Assuming that a sound system consists of one of each, how many different sound systems can she advertise?

4. Sherlock plans to read 4 cold case files from 15 that were submitted to him for review. He does not care what order he reads the files. How many ways can he do his reading?

5. Where in Pascal’s triangle would you find the number corresponding to C(10,3)?
6. How many 3-element subsets does set A have if \( n(A) = 9 \)?

7. In a “trifecta” wager, you must pick the 1\(^{st}\), 2\(^{nd}\), and 3\(^{rd}\) place winners in the exact order of their finish. How many trifectas are possible in a 12-horse race?

8. \( P(100,99) = \)

For problems 9-13, you are to determine the number of 3-digit numbers that can be formed using the digits in the set \( \{0, 1, 2, 3, 4\} \) and subject to the given conditions. The hundred’s digit cannot be 0.

9. Repetitions are allowed.

10. Repetitions are not allowed.

11. The numbers are even and repetitions are allowed.
12. The numbers are even and repetitions are not allowed.

13. The numbers are even, greater than 200, and repetitions are allowed.

Use the following information to answer questions 14 through 16. A UAM committee is to be formed to present a unified cell phone use policy on campus. The committee is to consist of 3 students and 3 faculty members. Twenty students and 7 faculty members volunteered to serve on this committee.

14. In how many ways can the student members on this committee be selected from the volunteers?

15. In how many ways can the faculty members on this committee be selected from the volunteers?

16. In how many ways can the members on this committee be selected from the volunteers?
17. Write out Pascal’s triangle to Row 6.

18. Mr. Lynde has 14 students in his Survey of Mathematics class this semester. He would like two of these students, Sabrina and Rex, to sit on the front row with a seat between them and with Sabrina always seated to Rex’s right. There are 7 desks on the front row and Mr. Lynde wants all of them to be occupied. Under these conditions, how many front-row arrangements are possible?

19. Suppose you flip a coin 3 times. Draw a tree diagram to illustrate all possible outcomes. List the outcomes to the right of the tree diagram as a series of H’s and T’s.
Unit 4 – Probability

Section 1 – Basic Probability

Random phenomena are occurrences that vary from case-to-case. Although we never know exactly how random phenomena will end, we can calculate a number called the *probability* that it will occur a certain way.

To calculate probability, you will need an *experiment* which is any observation of the random phenomena. The different possible results of the experiments are called *outcomes* and the set of all possible outcomes is called a *sample space*. Counting methods are very useful in constructing a sample space.

**Example 1:** Determine a sample space for each experiment.

a. The possible birth order in relation to gender for three children born to a family.

b. We roll two dice and observe the pair of numbers showing on the top faces.
In probability theory, an event is a subset of the sample space.

Example 2: If we consider the situations in example 1, determine the following events.

a. Two girls and one boy born to a family.

b. A total of 7 occurs when we roll a pair of dice.

Properties of Probability

The probability of an outcome in a sample space is a number between 0 and 1 inclusive. The sum of the probabilities of all the outcomes in a sample space must be 1.

The probability of an event $E$ is defined as the sum of the probabilities of the outcomes that make up $E$. It is written as $P(E)$.

In short, we have these basic properties of probability in a sample space $S$ in which $E$ is an event in $S$.

1. $0 \leq P(E) \leq 1$

2. $P(\emptyset) = 0$

3. $P(S) = 1$

4. To compute the probability of an event, $P(E) = \frac{n(E)}{n(S)}$.

Example 3: Use the answers from example 1 and 2 to answer the following:

a. The probability that two girls and one boy are born to a family.

b. The probability you will roll a total of 7 when you roll a pair of dice.
Using Counting Methods with Probability

Example 4: From the combination section, we know there are \( C(52,5) \) ways to select a 5-card hand from a 52-card deck. What is the probability of drawing only hearts for your 5-card hand.

Example 5: A committee of 10 people need to select two of their committee members to attend a conference. What is the probability that two of four friends on the committee are chosen to attend the conference?

Example 6: The table below summarizes a survey involving the relationship between living arrangements and grade point average for a group of students.

<table>
<thead>
<tr>
<th></th>
<th>On Campus</th>
<th>At Home</th>
<th>Apartment</th>
<th>Totals</th>
</tr>
</thead>
<tbody>
<tr>
<td>Below 2.0</td>
<td>102</td>
<td>31</td>
<td>57</td>
<td>190</td>
</tr>
<tr>
<td>2.0 to 3.0</td>
<td>76</td>
<td>23</td>
<td>31</td>
<td>130</td>
</tr>
<tr>
<td>Over 3.0</td>
<td>25</td>
<td>72</td>
<td>13</td>
<td>110</td>
</tr>
<tr>
<td>Totals</td>
<td>203</td>
<td>126</td>
<td>101</td>
<td>430</td>
</tr>
</tbody>
</table>

a. If we select a student randomly from this group, what is the probability the student will have a G.P.A. of at least 3.0?

b. If we select a student randomly who lives at home, what is the probability the student will have a G.P.A. below 2.0?
**Probability and Genetics**

A Punnett square is a table that summarizes the genetic possibilities of crossing two members of a species. An example of a Punnett square is below:

![Punnett Square Example]

**Example 7:** Sickle-cell anemia is a serious inherited disease. A person with two sickle-cell genes will have the disease but a person with only one sickle-cell gene will be a carrier of the disease.

\[ s = \text{sickle cell gene} \]
\[ n = \text{normal gene} \]

What would be the sickle-cell status of the children born to parents who are both carriers?

Determine \( P(\text{child has sickle-cell anemia}). \)
Example 8: In cross-breeding a certain flower, genetics has found that flower color does not dominate. For example, in these flowers, a flower with one blue gene and one white gene will have lavender flowers. If we cross two lavender flowers, draw a Punnett square that shows the results of crossing these two first generation plants.

\[
\begin{array}{c|c|c}
\text{Blue} & \text{White} & \text{Blue} \\
\hline
\text{Blue} & \text{Lavender} & \text{Blue} \\
\text{White} & \text{Lavender} & \text{White} \\
\end{array}
\]

What is the probability of getting blue flowers in second generation plants?

**Odds**

When you calculate odds of an event you compare the number of outcomes against the event and the number of outcomes for the event. There are two types of odds: odds against and odds in favor.

\[
\text{Odds Against: } n(E') : n(E)
\]

\[
\text{Odds in Favor: } n(E) : n(E')
\]

Alternate formulas for odds are

\[
\text{Odds Against: } \frac{P(E')}{P(E)}
\]

\[
\text{Odds in Favor: } \frac{P(E)}{P(E')}
\]

It is important to note that the odds in favor or against an event is not the same as the probability of the even happening or not happening. In fact, it can be stated that \(n(E') + n(E) = n(S)\) where \(E\) is an event in sample space, \(S\).
Example 9: If the odds in favor of an event are 2:5, what is the probability of the event happening?

Example 10: Reach into the following box and select one shape at random. Calculate the following.

a. $P(\text{heart shape})$

b. Odds in favor of selecting the heart shape

c. Odds against selecting the moon shape

d. $P(\text{lightening bolt})$

e. $P(\text{not selecting star shape})$

f. Odds in against selecting star shape
Section 2 – Complements and Unions of Events

Complement of an Event
To compute the probability of the complement of an event, then

\[ P(E') = 1 - P(E). \]

Example 1: The table below summarizes a survey involving the relationship between living arrangements and grade point average for a group of students.

<table>
<thead>
<tr>
<th></th>
<th>On Campus</th>
<th>At Home</th>
<th>Apartment</th>
<th>Totals</th>
</tr>
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<td>101</td>
<td>430</td>
</tr>
</tbody>
</table>

a. Let \( E \) be the event that a student lives at home. Determine \( P(E) \).

b. Determine \( P(E') \).

Union of Events
In probability, we join events using the set operations union and intersection. Unions represent combining two events together and involve the operations of addition and subtraction.

To compute the union of the probability of two events, \( E \) and \( F \),

\[ P(E \cup F) = P(E) + P(F) - P(E \cap F). \]

If \( E \) and \( F \) have no outcomes in common, they are called mutually exclusive events. In this case, \( E \cap F = \emptyset \).
Example 2: Use the Venn diagram below to calculate the indicated probabilities.

\[
\begin{array}{ccc}
S & P(A) & P(B) \\
0.16 & 0.24 & 0.18 \\
& 0.42 & \\
\end{array}
\]

\[P(A) = \]
\[P(B) = \]
\[P(A \cup B) = \]

Example 3: If we select a single card from a standard 52-card deck, what is the probability that we either draw a heart or a face card?

Example 4: If \(P(A \cup B) = 0.76\), \(P(A) = 0.41\) and \(P(B) = 0.51\), determine \(P(A \cap B)\).

Example 5: If \(P(A \cup B) = 0.74\), \(P(B) = 0.48\), and \(P(A \cap B) = 0.38\), find \(P(A)\).
Example 6: The table below summarizes a survey involving the relationship between living arrangements and grade point average for a group of students.

<table>
<thead>
<tr>
<th></th>
<th>On Campus</th>
<th>At Home</th>
<th>Apartment</th>
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<td>203</td>
<td>126</td>
<td>101</td>
<td>430</td>
</tr>
</tbody>
</table>

If a student from this group is selected at random, calculate the following

a. $P$ (student lives on campus)

b. $P$ (student does not live on campus)

c. $P$ (student has at least a 2.0 G.P.A. or lives off campus)

d. $P$ (student lives at home or has a G.P.A. over 3.0)

e. Odds in favor of the student not living on campus.
Section 3 – Conditional Probability and Intersections

Conditional Probability

When we calculate the probability of an event \( A \) assuming that event \( B \) has already occurred, we call this the conditional probability of \( A \), given \( B \).

We denote conditional probability by \( P(A|B) \). This is read “probability of \( A \) given that \( B \) has occurred.” This type of probability means you will calculate a probability knowing that something else has already happened.

Example 1: Assume we roll two dice and the total showing is greater than 8. What is the probability that the total is even?

<table>
<thead>
<tr>
<th>Die Pips</th>
<th>1</th>
<th>2</th>
<th>3</th>
<th>4</th>
<th>5</th>
<th>6</th>
</tr>
</thead>
<tbody>
<tr>
<td>1</td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td>2</td>
<td></td>
<td></td>
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<tr>
<td>3</td>
<td></td>
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<tr>
<td>4</td>
<td></td>
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</tr>
<tr>
<td>5</td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td>6</td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
</tr>
</tbody>
</table>
Mathematically, conditional probability can be determined with this formula:

\[ P(A|B) = \frac{P(A \cap B)}{P(B)} \]

and from this we can say an intersection of two events can be found with

\[ P(A \cap B) = P(B) \times P(A|B) \]

**Example 2:** Determine the following from the given Venn diagram:

a. \( P(A|B) = \)

b. \( P(B|A) = \)

---

**Independent and Dependent Events**

Dependent events are events that influence each other as the situation progresses. In a Venn diagram, dependent events are represented as having intersections that are not empty sets. For example, events A and B in example 2 are dependent events since \( P(A \cap B) \neq \emptyset \).

Mathematically, we can state

If \( P(E) = P(E|F) \), then the events \( E \) and \( F \) are independent events.

and

If \( P(E) \neq P(E|F) \) then the events \( E \) and \( F \) are dependent events.
**Example 3:** Suppose that a card is drawn at random from the following special deck of cards. Let the following represent events:

- **H** = the card drawn is a heart
- **A** = the card drawn is an ace
- **F** = the card drawn is a face card (i.e. jack, queen, king)

a. \( P(H) = \)

b. \( P(F) = \)

c. \( P(H \cap A) = \)

d. \( P(H | F) = \)

e. \( P(A | F) = \)

f. \( P(H | A) = \)

g. \( P(A | H) = \)

**Example 4:** Using the card deck and events from example 3, determine the following:

a. Are events \( H \) and \( F \) independent?

b. Are events \( A \) and \( F \) independent?
Example 5: The following data represents the results from a survey of students involving political parties and their choice for president in the 2016 election.

<table>
<thead>
<tr>
<th></th>
<th>Democrat</th>
<th>Republican</th>
<th>Independent</th>
<th>Totals</th>
</tr>
</thead>
<tbody>
<tr>
<td>Prefer H. Clinton</td>
<td>98</td>
<td>16</td>
<td>12</td>
<td>126</td>
</tr>
<tr>
<td>Prefer Paul Ryan</td>
<td>22</td>
<td>64</td>
<td>32</td>
<td>118</td>
</tr>
<tr>
<td>Totals</td>
<td>120</td>
<td>80</td>
<td>44</td>
<td>244</td>
</tr>
</tbody>
</table>

Determine the following:

a. \(P(\text{student is Republican and he prefers Ryan})\)

b. \(P(\text{student prefers Ryan given he is Republican})\)

c. \(P(\text{student is Republican given he prefers Ryan})\)

d. \(P(\text{student is Democrat and prefers Clinton})\)

e. \(P(\text{student is Democrat or prefers Clinton})\)

f. \(P(\text{student prefers Clinton given he is an Independent})\)

g. Are \(P(\text{prefers Clinton})\) and \(P(\text{Democrat})\) independent events?
**Conditional Probability and Tree Diagrams**

We can represent an experiment that happens in stages with a tree whose branches represent the outcomes of the experiment. We will call these trees *probability trees*.

To determine the probabilities, recall that each branch of a tree diagram happens as the result of a previous event. So, the secondary branches of a *probability tree* represents *conditional probability* as illustrated in the figure below:

\[
P(C | A) = P(A) \times P(C | A) \\
P(D | A) = P(A) \times P(D | A) \\
P(C | B) = P(B) \times P(C | B) \\
P(D | B) = P(B) \times P(D | B)
\]

\[
P(C) = P(C \cap A) + P(C \cap B) \\
P(D) = P(D \cap A) + P(D \cap B)
\]
Example 6 Compute the following from the given probability tree. Round your answers to two decimal places.

a. \( P(C \cap A) = \)

b. \( P(D \cap A) = \)

c. \( P(C \cap B) = \)

d. \( P(D \cap B) = \)

e. \( P(C) = \)

f. \( P(D) = \)

g. \( P(C|A) = \)

h. \( P(D|A) = \)

i. \( P(C|B) = \)

j. \( P(B|C) = \)

k. \( P(A|D) = \)

l. \( P(A|C) = \)
Example 7: Maddie is taking part in a lottery for a room in one of the UAM on-campus housing options. She is guaranteed a space, but she will have to draw a card randomly to determine exactly which room she will have. Each card will have the name of the housing option – Maxwell (M), Bankston (B), or Horsfall (H) – and whether she will have a single (S) or double (D) occupancy room. 33% of the available space is in Maxwell, 15% in Bankston, and 52% in Horsfall. 75% of the available rooms in Maxwell and Bankston and 50% of the available rooms in Horsfall are double occupancy.

a. Draw a probability tree of this situation.

b. Determine $P(Maddie \text{ drew a single room})$

c. Determine $P(Maddie \text{ drew a double room})$

d. Determine $P(Maddie \text{ has a single room given she drew Bankston})$

e. Determine $P(Maddie \text{ drew Bankston given she has a single room})$
**With and Without Replacement**

When working with probability problems that involve picking objects randomly from a given set, understanding “with” and “without” replacement is critical to computing probabilities correctly. When there is a “with” replacement, the total number of objects remains the same for each blind draw. However, “without” replacement results in changes in the count of objects in the situation. In fact, “without” replacement creates a *conditional probability*. Consider the following:

**Example 8:** A box contains 8 tan socks and 12 black socks. Two socks are drawn from this box *without* replacement. Calculate the following:

a. $P(\text{both socks are tan}) = P(\text{tan and tan}) =$

b. $P(\text{both socks are black}) = P(\text{black and black}) =$

c. $P(\text{two different colors}) = P(\text{Black and Tan or Tan and Black}) =$

**Example 9:** A box contains 8 tan socks and 12 black socks. Two socks are drawn from this box *with* replacement. Calculate the following probabilities:

a. $P(\text{both socks are tan}) = P(\text{tan and tan}) =$

b. $P(\text{both socks are black}) = P(\text{black and black}) =$

c. $P(\text{two different colors}) = P(\text{Black and Tan or Tan and Black}) =$
**Section 4 – Expected Value**

Expected value is a method by which we can analyze strategies in decision-making. The definition of expected value is

If an experiment has \( n \) outcomes with probabilities \( P_1, P_2, P_3, ..., P_n \) and values associated with each of those probabilities \( V_1, V_2, V_3, ..., V_n \), then the *expected value* of the experiment is

\[
e.v. = (P_1 \cdot V_1) + (P_2 \cdot V_2) + (P_3 \cdot V_3) + \cdots + (P_n \cdot V_n).
\]

In evaluating games of chance, when \( e.v. = 0 \), the game is considered *fair*.

To compute expected value, you must always consider the value of each probability AND how much it costs to “play the game.”

**Example 1:** In a casino, a simple roulette game allows for you to place a $1 bet on one of the 38 numbers on the wheel. If your number comes up, the casino will give you $30 (you also get to keep your $1 bet) otherwise you will lose the $1 bet. What is the expected value of the bet? Is this a fair game?

**Hint:** To calculate \( e.v. \) of this example, you have two outcomes:

1. You win and the value to you is +$30
2. You lose and the value to you is − $1
Example 2: At a recent carnival, a game was provided in which four types of shapes were placed in a box and prizes were given according to which shape the player pulled out of the box. To play the game, the player had to pay $1.50 in which they did not get back after the game was over. The prizes were:

- $2.00 if you picked a heart shape,
- $3.00 if you picked a star shape,
- $4.00 if you picked a moon shape,
- $5.00 if you picked the lightening bolt.

Calculate the expected value of the game and determine if it is fair. If the game is not fair, to whose advantage does the expected value of the game favor?

Example 3: Assume that it costs $2 to play a state’s daily number lottery. The player chooses a three digit number between 000 and 999, inclusive, and if the number is selected that day, then the player wins $500 (this means the player’s profit is $500 − $2 = $498). What is the expected value of this game? What should the price of the ticket be in order to make this game fair?
Example 4: A student is taking a multiple choice test consisting of 5 choices for each question. The student earns 1 point for each correct answer and ½ a point is subtracted for each incorrect answer. Questions left blank neither receive nor lose points.

a. Determine the expected value of randomly guessing an answer to the question. Is it to the student’s advantage to guess?

b. Suppose the student can eliminate two of the five choices. What is the new expected value? Is it to the student’s advantage to guess, now?

Example 5: Suppose you operate a donut shop on campus. It costs you $0.75 to make each donut and you sell them for $1.00 each. If you have extra donuts at the end of the day, you donate them to a homeless shelter and do not ask for a receipt. Suppose the past ten days demand for your donuts is as follows:

<table>
<thead>
<tr>
<th>Demand for Donuts</th>
<th>1500</th>
<th>1050</th>
</tr>
</thead>
<tbody>
<tr>
<td># of days for these sales</td>
<td>4</td>
<td>6</td>
</tr>
</tbody>
</table>

Determine your expected value for your profit or loss if you back 1500 donuts for tomorrow morning.
Probability Review

1. An experiment consists of flipping three coins. The outcomes of the experiments are represented by strings of T’s and H’s such as THT and TTT. How many elements are in the sample space of this experiment?

2. If a pair of fair dice is rolled, what is the probability that the sum of the pips (dots) is 6?

3. One card is selected at random from a standard 52-card deck. What are the odds against drawing a spade?

4. The odds against Franks on Fire winning the race at Oaklawn are 1 to 5. What was the probability that Franks on Fire will win the race?
5. Use the following Punnett square to determine the probability of getting white snap-dragons when two second-generation snapdragons are crossed. (ww = white, RR = red, and Rw and wR = pink)

<table>
<thead>
<tr>
<th>First Parent</th>
<th>R</th>
<th>w</th>
</tr>
</thead>
<tbody>
<tr>
<td>R</td>
<td>RR</td>
<td>Rw</td>
</tr>
<tr>
<td>w</td>
<td>wR</td>
<td>ww</td>
</tr>
</tbody>
</table>

6. If the probability that you will pass this test is 72%, what is the probability that you will not pass this test?

7. What is the probability of getting either an ace or a club when drawing a single card from a standard 52-card deck?

8. If \( P(A \cup B) = 0.7 \), \( P(A) = 0.4 \), and \( P(A \cap B) = 0.2 \), then what is \( P(B) \)?
For problems 9 through 11, use the following chart to calculate the indicated probability.

<table>
<thead>
<tr>
<th>Teacher</th>
<th>Female Students</th>
<th>Male Students</th>
<th>Totals</th>
</tr>
</thead>
<tbody>
<tr>
<td>Chapman</td>
<td>23</td>
<td>7</td>
<td>30</td>
</tr>
<tr>
<td>Lynde</td>
<td>18</td>
<td>6</td>
<td>24</td>
</tr>
<tr>
<td>Sayyar</td>
<td>19</td>
<td>3</td>
<td>22</td>
</tr>
<tr>
<td>Totals</td>
<td>60</td>
<td>16</td>
<td>76</td>
</tr>
</tbody>
</table>

9. \( P(\text{the student was female and in Mr. Lynde's class}) = \)

10. \( P(\text{the student was female or in Mr. Lynde's class}) = \)

11. \( P(\text{the student was male| the student was in Dr. Sayyar's class}) = \)

Use the tree diagram shown below to calculate the indicated probabilities in problems 12-14. Round your answer to two decimal places.

12. \( P(D) = \)

13. \( P(C|B) = \)

14. \( P(B|D) = \)
15. Suppose a box has cards with numbers on it as shown below.

Let $E =$ an even number is drawn from the box and 
$F =$ a number greater than 4 is drawn from the box.

Are $E$ and $F$ independent events? Explain your answer.

16. Suppose that 100 tickets are sold at $5 each for a raffle. There is a grand prize of $300 and two second prizes of $150.

A) Calculate the expected value for a ticket in this raffle.

B) Is this raffle a fair game? If so, explain why. If not, what price should be charged for a raffle ticket to make it a fair game?
17. An experiment consists of drawing 2 colored blocks \textit{without replacement} from an envelope that contains 5 red blocks, 3 blue blocks, and 2 white blocks. Calculate the following probabilities.

A) \( P(\text{both blocks drawn are red}) = \)

B) \( P(\text{one block drawn is blue and one block drawn is white}) = \)

18. Draw and label a tree diagram to represent the following situation. Then use your tree diagram to calculate the indicated probabilities (to 4 decimal places).

\textit{You want to purchase a DVD drive for your laptop computer. Assume 65\% of the drives are made outside of the United States. Of the U.S.-made drives, 5\% are defective; of the foreign-made drives, 17\% are defective.}

Determine the following:

\( P(\text{the drive is defective}) \)

\( P(\text{the drive was foreign-made given that the drive was defective}) \)
Unit 5 – Statistics

Section 1 – Organizing and Visualizing Data

Statistics is an area of mathematics in which we are interested in gathering, organizing, analyzing, and making predictions from numerical information (data).

The gathering of data can be accomplished by administering surveys, monitoring responses to events, and observations. A population is made up of the entities generating the data and a sample is a subset of the population. A sample should reflect the population as a whole and should not be corrupted by bias. Examples of bias include selection bias in which members of the population are selected with pre-determined characteristics that do not represent the population in general and leading-question bias in surveys in which surveyors ask questions that guide the responses toward a desired end.

The organization and visualization of data is easily accomplished with the use of charts and graphs. In this chapter, we will use frequency distributions, relative frequency distributions, stem-and-leaf plots, bar graphs, histograms, box-and-whisker plots, normal distribution graphs, and linear regressions to organize and visualize data.

Frequency Distributions

A frequency distribution is a tally of the number of occurrences (the frequency) in which a particular data point appears in data.

A relative frequency distribution details the percent of occurrences in which a particular data point appears in data.

A stem-and-leaf plot is a more sophisticated version of a frequency distribution in which the data points are organized around a given place value.
**Example 1:** The following represents the number of miles per gallon a random sampling of 30 hybrid vehicles obtain in city driving.

```
38  45  27  36  48  24  31  36  51  48
24  38  53  34  46  48  51  27  29  45
46  34  27  48  29  53  36  51  45  38
```

Construct the following for the data:

a. A frequency distribution

b. A relative frequency distribution

c. A stem-and-leaf plot

---

**Bar Graphs and Histograms**

A *bar graph* is a visualization of a frequency distribution. When the frequency distribution represents continuous distributions or contains a large number of possible values, a bar graph becomes a *histogram*. In drawing bar graphs, the categories are specified along the horizontal axis and the frequency along the vertical axis. A histogram does not allow spaces between the bars above each category.
Many graphing calculators have a statistics application that can be utilized to construct bar graphs and histograms.

**Example 2:** The following chart represents the ACT scores for entering freshmen at a local university from 2010 – 2012. Draw a bar graph to represent this data.

<table>
<thead>
<tr>
<th>Score</th>
<th>Frequency</th>
</tr>
</thead>
<tbody>
<tr>
<td>14</td>
<td>6</td>
</tr>
<tr>
<td>15</td>
<td>10</td>
</tr>
<tr>
<td>16</td>
<td>9</td>
</tr>
<tr>
<td>17</td>
<td>18</td>
</tr>
<tr>
<td>18</td>
<td>21</td>
</tr>
<tr>
<td>19</td>
<td>25</td>
</tr>
<tr>
<td>20</td>
<td>20</td>
</tr>
<tr>
<td>21</td>
<td>27</td>
</tr>
<tr>
<td>22</td>
<td>23</td>
</tr>
</tbody>
</table>

**Example 3:** The following represents the number of days a sample of students missed their Survey of Mathematics class during the Spring 2014 semester. Construct a frequency distribution with 4 classes to represent the data. Draw a histogram to represent the data.

<table>
<thead>
<tr>
<th>0</th>
<th>4</th>
<th>12</th>
<th>24</th>
<th>5</th>
<th>1</th>
<th>7</th>
<th>31</th>
<th>14</th>
<th>0</th>
<th>8</th>
<th>16</th>
</tr>
</thead>
<tbody>
<tr>
<td>1</td>
<td>32</td>
<td>11</td>
<td>12</td>
<td>0</td>
<td>15</td>
<td>21</td>
<td>37</td>
<td>22</td>
<td>39</td>
<td>2</td>
<td>3</td>
</tr>
<tr>
<td>3</td>
<td>6</td>
<td>10</td>
<td>1</td>
<td>25</td>
<td>4</td>
<td>20</td>
<td>39</td>
<td>0</td>
<td>1</td>
<td>11</td>
<td>2</td>
</tr>
</tbody>
</table>
Back-to-back stem-and-leaf plots and double bar graphs are common methods used to organize and visualize two sets of data with similar categories.

**Example 4:** The following represent scores made on Test 3 and Test 4 during the Spring 2014 semester in Survey of Mathematics. Construct a back-to-back stem-and-leaf plots and a double bar graph to display the data.

<table>
<thead>
<tr>
<th>Test 3:</th>
<th>66</th>
<th>81</th>
<th>75</th>
<th>41</th>
<th>56</th>
<th>91</th>
<th>65</th>
<th>82</th>
<th>98</th>
<th>59</th>
<th>93</th>
<th>42</th>
</tr>
</thead>
<tbody>
<tr>
<td></td>
<td>62</td>
<td>90</td>
<td>72</td>
<td>85</td>
<td>80</td>
<td>21</td>
<td>46</td>
<td>79</td>
<td>88</td>
<td>51</td>
<td>77</td>
<td>96</td>
</tr>
<tr>
<td>Test 4:</td>
<td>68</td>
<td>72</td>
<td>90</td>
<td>89</td>
<td>85</td>
<td>66</td>
<td>70</td>
<td>64</td>
<td>60</td>
<td>88</td>
<td>53</td>
<td>75</td>
</tr>
<tr>
<td></td>
<td>79</td>
<td>60</td>
<td>92</td>
<td>94</td>
<td>52</td>
<td>77</td>
<td>91</td>
<td>80</td>
<td>71</td>
<td>75</td>
<td>72</td>
<td></td>
</tr>
</tbody>
</table>
Section 2 – Measures of Central Tendency

Measurements of Data

Mean, median, mode, and range are common measurements used to help with data analysis. The mean of a data set is commonly referred to as an average and is computed by dividing the sum of the data points by the number of data points. Mathematically, we say

$$\bar{x} = \frac{x_1 + x_2 + x_3 + \cdots + x_n}{n}$$

where the data set has $n$ points of data, $x_i$ represents the individual data points, and $\bar{x}$ is the mean. The sigma symbol, $\Sigma$, can be used to shorten the numerator of the mean calculation by

$$\bar{x} = \frac{x_1 + x_2 + x_3 + \cdots + x_n}{n} = \frac{\Sigma x}{n}.$$  

The median is the middle value in a list of numbers that is arranged in increasing (or decreasing) order.

1. If there is an odd number of values, then the median is the number in the middle position.
2. If there is an even number of values, then the median is the average of the two middle numbers.

The mode of a data set is the data point that occurs the most frequently. If two items occur with the most frequency, then each is a mode. If more than two scores occur most frequently, then we will rule there is not mode.

The range of a data set is the difference between the largest and smallest data values in the set.

The mean, median, and mode of a data set are three measures of central tendency.
**Example 1:** The following represents the number of days a sample of students missed their Survey of Mathematics class during the Spring 2014 semester. Determine the mean, median, mode, and range of the data. Of the four measurements, which would be the best to use to try to persuade your instructor to award bonus points for attendance?

0 4 12 24 5 1 7 31 14 0 8 16
1 32 11 12 0 15 21 37 22 39 2 3
3 6 10 1 25 4 20 39 0 1 11 2

**Example 2:** The following bar graph represents grades on a recent Survey of Mathematics quiz. Determine the mean, median, mode, and range of the data.

If one more student took the quiz and made a 9, how would this affect the mean?
Example 3: In order to get a C in Survey of Mathematics your average must be at least 69.5 since your instructor will round to the nearest whole number. Also, the grade on the final exam replaces the lowest test or your quiz average provided that it is higher than the grade it replaces and no test was missed. If a test is missed, the final will replace the missed grade but no other.

Suppose a student has the following grades: Test 1 (55), Test 2 (68), Test 3 (75), Test 4 (81), and quiz average (69). What is the least grade the student can make on the final and get a C in the class? Would it be possible for the student to make a B in the course? Why or why not?

Five Number Summary

The five-number summary of a data set is comprised of

Minimum, First Quartile (Q1), Median, Third Quartile (Q3), Maximum

in which the median divides the data set into two halves. The set of numbers below the median is the lower half and the set of data above the median is the upper half. The first quartile, abbreviated Q1, is the median of the lower half while the third quartile, Q3, is the median of the upper half. The five-number summary is represented by a graph called a box-and-whisker.
Example 4: Determine the five-number summary of the following data set and represent it in a box-and-whisker graph.

12 32 11 12 4 15 21 37 22 39 22 3
13
3 6 10 18 25 4 20 39 24 12 11 21
Section 3 – Measures of Dispersion

Two measures of dispersion are range and standard deviation. Recall, range is the distance from the minimum value to the maximum value in a data set.

Standard deviation

Standard deviation is a measure of dispersion based on calculating the distance of each data value from the mean. A relatively small standard deviation indicates that the data values are not too spread out. To calculate the standard deviation of a sample:

1. Calculate the mean, $\bar{x}$

2. Calculate the deviation from the mean for each data point, which is 
   
   $x - \bar{x}$, where $x$ is the data point and $\bar{x}$ is the mean of the data

3. Calculate $(x - \bar{x})^2$ for each data point (score) $x$

4. Determine the variance of the data, which is
   
   $\frac{\sum(x - \bar{x})^2}{n}$

   in which $n$ is the number of data points in the data set.

5. Take the square root of the variance to determine the standard deviation:
   
   $s_x = \sqrt{\frac{\sum(x - \bar{x})^2}{n - 1}}$

Of course, most graphing calculators have a statistic application which will calculate the standard deviation of a data set.
**Example 1:** Complete the following chart to find the standard deviation for the given set of data:

<table>
<thead>
<tr>
<th>Data Value (Score)</th>
<th>Deviation from the mean, ((x - \bar{x}))</th>
<th>Deviation squared, ((x - \bar{x})^2)</th>
</tr>
</thead>
<tbody>
<tr>
<td>0</td>
<td></td>
<td></td>
</tr>
<tr>
<td>2</td>
<td></td>
<td></td>
</tr>
<tr>
<td>1</td>
<td></td>
<td></td>
</tr>
<tr>
<td>4</td>
<td></td>
<td></td>
</tr>
<tr>
<td>2</td>
<td></td>
<td></td>
</tr>
<tr>
<td>3</td>
<td></td>
<td></td>
</tr>
</tbody>
</table>

\[
\frac{\sum(x - \bar{x})^2}{n} =
\]

\[
s_x = \sqrt{\frac{\sum(x - \bar{x})^2}{n - 1}} =
\]
Example 2: The following chart represents the ACT scores for entering freshmen at a local university from 2010 – 2012.

<table>
<thead>
<tr>
<th>Score</th>
<th>14</th>
<th>15</th>
<th>16</th>
<th>17</th>
<th>18</th>
<th>19</th>
<th>20</th>
<th>21</th>
<th>22</th>
</tr>
</thead>
<tbody>
<tr>
<td>Frequency</td>
<td>6</td>
<td>10</td>
<td>9</td>
<td>18</td>
<td>21</td>
<td>25</td>
<td>20</td>
<td>27</td>
<td>23</td>
</tr>
</tbody>
</table>

Use the above frequency table to calculate standard deviation of the data:

Coefficient of Variation

In order to use the standard deviation to compare different sets of data, we must first make the two standard deviations comparable. We do this by determining the coefficient of variation:

For a data set with mean $\bar{x}$ and standard deviation $s_x$, the coefficient of variation, denoted as $CV$, is computed by

$$CV = \frac{s_x}{\bar{x}} \cdot 100\%.$$

Example 3: The following represent scores made on Test 3 and Test 4 during the Spring 2014 semester in Survey of Mathematics. Use the coefficient of variation to compare the variation of the two data sets.

Test 3: 66 81 75 41 56 91 65 82 98 59 93 22
Test 4: 68 72 90 89 85 66 70 64 60 88 53
Section 4 – The Normal Distribution

Many real-life data sets are described by the most common distribution in statistics: the normal distribution. The graph of a normal distribution is often referred to as a bell-shaped curve.

The mean, median, and mode of a normal distribution are the same and the curve is symmetric with respect to the mean. Finally, the area under a normal curve equals 1. Since the normal curve is symmetric with respect to the mean, 50% of the area under the curve is located on each side of the mean.

An inflection point is a point on a curve where the curve changes from being concave up to being concave down. For a normal curve, inflection points are located one standard deviation from the mean.

68-95-99.7 rule

Approximately 68% of the data values occur within 1 standard deviation of the mean, 95% of the data lies within 2 standard deviations, and 99.7% of the data lies within 3 standard deviations in a normal distribution. The 68-95-99.7 rule can be used to estimate the number of scores we expect to fall within 1, 2, or 3 standard deviations of the mean of a normal distribution.
Example 1: Suppose that the distribution of scores of 100 students who take a standardized academic test is a normal distribution. If the distribution’s mean is 45 and its standard deviation is 3,

a. how many scores do we expect to fall between 42 and 48?

b. how many scores do we expect to fall above 51?

Example 2: Suppose that for a certain set of data, the mean is 15 and the standard deviation is 2. Determine the raw score associated with each of the following z-scores:

a. 1  raw score =

b. 2.5  raw score =

c. −4  raw score =
Example 3: Suppose that for a certain set of data, the mean is 40 and the standard deviation is 6. Determine the z-score associated with each of the following raw scores.

a. 70 \( z\)-score = ____________

b. 40 \( z\)-score = ____________

c. 10 \( z\)-score = ____________

Example 4: Determine the standard deviation if the \( z\)-score of a data point is 2.1, the raw score is 8, and the mean is 5.6.

Example 5: Determine the area under the normal curve from \( z = -1 \) to \( z = 1 \). This is denoted by the expression \( P(-1 \leq z \leq 1) \). Shade this region in the following graph:
Example 6: Suppose that women’s shoe sizes is normally distributed with a mean of 7 ½ and a standard deviation of 1. Approximately how many women out of a group of 750 would you expect to have shoe sizes between 5 ½ and 9 ½?

To calculate the percentage of a population that corresponds to a certain range of standard deviations with your calculator, you can use the calculate menu under the graphing option on a graphing calculator.

Enter the following into the $Y = \text{editor}$ on your calculator: $Y = \frac{e^{\frac{x^2}{2}}}{\sqrt{2\pi}}$.

Change the window on your calculator to $x\text{-min} = -5$ $y\text{-min} = -0.1$
$x\text{-max} = 5$ $y\text{-max} = 0.5$
$x\text{Scl} = 1$ $y\text{Scl} = 0.1$

Press the 2nd, Trace buttons
Choose option 7
Enter lower bound
Enter upper bound
Enter
Example 7: Using your graphing calculator, calculate the following for the normal distribution curve:

a. \( P(-2.5 \leq z \leq 1.8) \)

b. \( P(z \geq 0.75) \)

c. \( P(z \leq 0.75) \)
Example 8: Suppose that the averages for a class are normally distributed with a mean of 71 and a standard deviation of 4. Calculate the following. Label the normal distribution curve with both the raw scores and the z-scores.

a. The probability that a person selected at random will have an average between 76 and 90.

![Graph showing normal distribution with mean 71 and standard deviation 4.]

b. The probability that a person selected at random will have an average less than 60.

![Graph showing normal distribution with mean 71 and standard deviation 4.]
Section 5 – Linear Regression

When looking for a relationship between two sets of data, determine if there is a connection between the first data set and the second data set. This connection is referred to as a correlation. One way to visually examine two data sets and determine if there is a correlation is with a scatterplot. A scatterplot is a graph of the data points plotted on the $x, y$ coordinate plane.

Linear correlation exists if the data points graphed in a scatterplot tend to lie in a straight line. Scatterplots can have a positive linear correlation, a negative linear correlation, strong and weak variations of positive and negative linear correlations, and no linear correlation.

Example 1: Determine if the linear correlation of the following scatterplots:
A *line of best fit* is a line that minimizes the vertical distance from data points to a line. In this class, sketching a line of best fit will be accomplished by hand (usually by estimating vertical distances) and with a Linear Regression feature on most graphing calculators.

**Example 2:** Sketch the line of best fit for the following data:

<table>
<thead>
<tr>
<th>Hours doing homework</th>
<th>0</th>
<th>0.5</th>
<th>2</th>
<th>2.5</th>
<th>3.5</th>
<th>4</th>
<th>6.5</th>
<th>9</th>
<th>10.5</th>
</tr>
</thead>
<tbody>
<tr>
<td>Test 1 grade for class</td>
<td>22</td>
<td>28</td>
<td>46</td>
<td>52</td>
<td>63</td>
<td>70</td>
<td>81</td>
<td>85</td>
<td>95</td>
</tr>
</tbody>
</table>

![Graph](image)

**Hours Doing Homework**

a. What is the linear correlation of the data?

b. Using the Linear Regression function on your calculator, plot the data and line of best fit. How does it compare to the graph above?
When using the Linear Regression function on your calculator, you receive far more results than just the line of best fit. For now, we will focus on the linear correlation coefficient (i.e. the $r$ value) for a set of data.

The $r$ value will range from $-1$ to $1$ and indicates to what degree the data lies along a straight line. A $r$ value of $-1$ indicates a strong negative correlation, a $r$ value of $0$ indicates no linear correlation, and a $r$ value of $1$ indicates a strong positive linear correlation. The higher the absolute value of the $r$ value, the more linear the data set. The $r$ value is calculated with the following formula:

$$r = \frac{n \sum xy - (\sum x)(\sum y)}{\sqrt{n(\sum x^2) - (\sum x)^2} \sqrt{n(\sum y^2) - (\sum y)^2}}.$$

Additionally, we can use the $r$ value to determine what level of confidence we have in making predictions from the data. The confidence level is calculated using the number of data points in the set and the $r$ value. For the sake of brevity, we have included a confidence level table for data sets with 4 to 20 data points:

<table>
<thead>
<tr>
<th>$n$</th>
<th>$\alpha = 0.5$</th>
<th>$\alpha = 0.1$</th>
</tr>
</thead>
<tbody>
<tr>
<td>4</td>
<td>0.950</td>
<td>0.999</td>
</tr>
<tr>
<td>5</td>
<td>0.878</td>
<td>0.959</td>
</tr>
<tr>
<td>6</td>
<td>0.811</td>
<td>0.917</td>
</tr>
<tr>
<td>7</td>
<td>0.754</td>
<td>0.875</td>
</tr>
<tr>
<td>8</td>
<td>0.707</td>
<td>0.834</td>
</tr>
<tr>
<td>9</td>
<td>0.666</td>
<td>0.798</td>
</tr>
<tr>
<td>10</td>
<td>0.632</td>
<td>0.765</td>
</tr>
<tr>
<td>15</td>
<td>0.514</td>
<td>0.641</td>
</tr>
<tr>
<td>16</td>
<td>0.497</td>
<td>0.623</td>
</tr>
<tr>
<td>17</td>
<td>0.482</td>
<td>0.606</td>
</tr>
<tr>
<td>18</td>
<td>0.468</td>
<td>0.590</td>
</tr>
<tr>
<td>19</td>
<td>0.456</td>
<td>0.575</td>
</tr>
<tr>
<td>20</td>
<td>0.444</td>
<td>0.561</td>
</tr>
</tbody>
</table>

To determine your confidence level for using the data to make predictions:

1. Compute the $r$ value for $n$ data points
2. Consult line $n$ in the table
3. If $|r| >$ value in the column labeled $\alpha = 0.5$ on line $n$, then the confidence level is 95%.
4. If $|r| >$ value in the column labeled $\alpha = 0.1$ on line $n$, then the confidence level is 99%.

122
Example 3: For the data in Example 2, calculate the

a. $r$ value = ______________

b. Confidence level = ______________

c. Equation of line of best fit = __________________

d. The predicted grade for a student who spent 3 hours doing homework for the class. Use your line of best fit to determine this.

e. The predicted time spent studying for a student who has a 98% in the class. Use your line of best fit to determine this.
Example 4: A teacher randomly selected ten of her students to determine if there is any correlation between hours spent on social media per week and the grade in the class. The data collected is shown below:

<table>
<thead>
<tr>
<th>Hours spent on social media</th>
<th>21</th>
<th>50</th>
<th>36</th>
<th>53</th>
<th>66</th>
<th>79</th>
<th>31</th>
<th>14</th>
<th>89</th>
<th>41</th>
</tr>
</thead>
<tbody>
<tr>
<td>Grade in class</td>
<td>81</td>
<td>55</td>
<td>76</td>
<td>80</td>
<td>62</td>
<td>58</td>
<td>83</td>
<td>94</td>
<td>44</td>
<td>77</td>
</tr>
</tbody>
</table>

a. Calculate the equation of the line of best fit. Round all numbers to three decimal places.

b. Use your line of best fit to estimate the grade of a student who spent 30 hours on social media per week.

c. Use your line of best fit to estimate the number of hours spent on social media per week for a student whose grade is 70.

d. Calculate the linear correlation coefficient (round to three decimal places).

e. What is the statistical significance of this study using the confidence level from the data?
Statistics Review

For problems 1-6, use the data below which are the batting averages for the starting 9 players on the Lucky Joe's Auto Shop softball team.

\[0.422 \quad 0.225 \quad 0.399 \quad 0.307 \quad 0.419\]
\[0.280 \quad 0.254 \quad 0.242 \quad 0.215\]

1. The mean is

2. The median is

3. The mode is

4. The third quartile is

5. The (sample) standard deviation is approximately

6. The range is
For problems 7, 8, and 9, consider the data in the frequency table below which represents the responses from a class of Survey of Mathematics students to the question, “How many hours each week did you study for this class?”

<table>
<thead>
<tr>
<th>Number of hours studied each week</th>
<th>0</th>
<th>1</th>
<th>2</th>
<th>3</th>
</tr>
</thead>
<tbody>
<tr>
<td>Frequency</td>
<td>3</td>
<td>4</td>
<td>2</td>
<td>5</td>
</tr>
</tbody>
</table>

7. The mean number of hours studied each week was approximately

8. The median number of hours studied each week was

9. The range of hours studied each week was

10. Which of the following best describes the mode?

   A) It is the middle number in a set of data.
   B) It is the middle number in an ordered set of data.
   C) It is a statistic that is always less than the mean.
   D) It is always one of the data.
   E) It is the number that occurs most frequently.

11. Suppose that there are 4 people in a car and their mean weight is 164 pounds. One person gets out of the car and the mean weight of the remaining passengers is now 180 pounds. Find the weight of the person who got out of the car.
For problems 12 and 13, consider a normal distribution with a mean of $10$ and a standard deviation of $1.2$.

12. The $z$-score for a raw score of $9.7$ is $\underline{1.2}$.

13. The raw score that has a $z$-score of $2.5$ is $\underline{16.25}$.

14. Suppose that a set of data is normally distributed with a mean of $20$. If the $z$-score of $15$ is $-2$ then the standard deviation for this set of data is $\underline{17.32}$.

15. Calculate: $P(z \geq 1.43) \approx \underline{.075}.$

16. Calculate: $P(z \leq -0.57) \approx \underline{.284}$.

17. Calculate: $P(-1.96 \leq z \leq -0.54) \approx \underline{.364}.$
For problems 18 and 19, consider the box-and-whiskers plot shown below that represents the scores made by 150 welders on a certification test.

18. The median score on this test was

19. The approximate number of welders who scored below 50 on this test was

20. The scatterplot shown indicates

A) Significant positive linear correlation between the variables
B) Moderate positive linear correlation between the variables
C) No correlation between the variables
D) Significant negative linear correlation between the variables
E) Moderate negative linear correlation between the variables
21. Consider the set of data shown below.

<table>
<thead>
<tr>
<th>Number of Absences (x)</th>
<th>2</th>
<th>1</th>
<th>0</th>
<th>7</th>
<th>1</th>
<th>8</th>
<th>3</th>
<th>4</th>
</tr>
</thead>
<tbody>
<tr>
<td>Grade on Final Exam (y)</td>
<td>86</td>
<td>83</td>
<td>96</td>
<td>60</td>
<td>90</td>
<td>55</td>
<td>72</td>
<td>70</td>
</tr>
</tbody>
</table>

A) Determine the linear correlation coefficient for the data. Round your answer to 3 decimal places.

B) Determine if we can be 95% or 99% confident that there is a significant linear correlation between the two variables. Write a sentence stating your conclusion.

C) Give the equation of the line of best fit. Round coefficients and constants to one decimal place.

D) Use the equation of the line of best fit to find the grade (rounded to the nearest whole number) on the final exam of a student who missed class 6 times.
A university finds the data on an admissions test for entering freshmen is normally distributed with a mean of 50 and a standard deviation of 8.

A) Calculate the z-score of a raw score of 48.

B) If the university admits any student who scores 50 or above, approximately what percent of the applicants are admitted? Label and shade the normal distribution curve shown below with BOTH z and x values appropriate for this problem. Shade appropriately. Solve this problem and state your conclusion in a complete sentence.
23. Use the bar graph shown below to answer the following questions.

![Bar Graph]

A) How many SUVs are represented by this graph?

B) Determine the mean mph for the SUVs represented by this graph.

24. Construct a stem-and-leaf graph for the following set of data which represents the number of victims of identity theft in a sample of 5 cities.

77, 84, 88, 22, 30, 28, 78, 15, 35, 20, 31, 50, 64, 53, 42, 58, 60, 67